P- ISSN 1991-8941 E-ISSN 2706-6703 Journal of University of Anbar for Pure Science (JUAPS) Open Access
2012,(6), (1) :61-63

Study of Groups with basic property

Dunia Alawi Jarwan

University of Anbar - College of Education for pure Sciences.

ARTICLE INFO

Received: 5 / 10 /2011
Accepted: 13 / 3 /2012
Available online: 30/10/2012
DOI: 10.37652/juaps.2012.63152

Keywords:
dependence ,
independence ,
algebraic structure ,
finite group.

ABSTRACT

The purpose of this paper is to study the concept of dependence , independence and the basis of some algebraic structure and give the definition of a finite group with basic property and study some of its basic properties.

Introduction

The general Algebra **A** is called Algebra with basic property If any sub algebra of **A** has minimal Generating set(basis)such that for any two basis have the same number of elements.

Many of Algebraic structures with basic property for example the linear vector space.
(P. Jones [4]) studied some of semigroups with basic property ,then (V. Shiryaev [7]) studied the sub semigroups which have unique basis.

In this paper we studied the concept of dependence, independence and basis .Also we studied the finite groups with basic property and called them B-groups ,finally we obtained the some properties of the groups with basic property as every groups with basic property is periodic ,quaziprimary and the image of any group with basic property by homomorphism group also form a group with basic property

2- Dependence Relation and Basic concept

Definition (2-1) [2]:-Let S be a set , the dependence relation on S is a rule which associates with each finite subset $X \subseteq S$ certain elements of S, said to be dependent on X, further ,the following conditions are satisfied:

D1. If $X=\{x_1, \ldots,x_n\}$, then each x_i is dependence on X.

D2. (Transitivity) if z is dependent on $\{y_1,\ldots, y_m\}$ and each y_i dependent on $\{x_1, \ldots,x_n\}$,then z is dependent on $\{x_1, \ldots,x_n\}$.

D3.(Exchange property) .If y is dependent on $\{x_1, \ldots,x_n\}$ .but not on $\{x_2, \ldots,x_n\}$ then x_1 is dependent on $\{y_1, x_2, \ldots,x_n\}$.

Example(2-1) [2]**:**Let (F,+,.) be a field , then the notion of linear dependence over this field satisfies the above conditions.

* Corresponding author at: University of Anbar - College of Education for pure Sciences.E-mail address: Dunia_alaja@yahoo.com

Definition(2-2) [2]**:-**Let S be a set with dependence relation we defined generated of X where $X \subseteq S$ is the set of all elements of S which dependent on X , and denoted by $<X>$ and we say $<X>$ generated by X.

Definition(2-3) [2] **:-** A finite X $=\{x_i;\ I \in I\}$ of elements of S is said to be independent if no x_i is dependent on the remaining member of X; otherwise X is dependent.

Definition(2-4) [6]**:-** An independent family which generated S is called a basis of S.

Proposition(2-1) [2]: Let S be a set with a dependence relation, then for condition are equivalent:

a)X is maximal independent subset of S.

b)X is a minimal generators set of S .

c) X is a base of S.

proof.

(a)↔(c) Let X be a maximal independent subset of S, then any $x \in X$ is dependent on X by (D1) ,and if $y \notin X$, then $X \cup \{y\}$, by maximality , so some element is dependent on the say $x \in X$ is dependent on $X' \cup\{y\}$, where X' is the complement of {x} in X. Since X is independent, x is not dependent on X'

Hence by (D2) y is dependent on $X= X' \cup\{x\}$,and this show that X spans S.

Conversely ,if X is basis, it is independent , but every element in X is dependent on it, so X is maximal independent.

(b)↔(c) .Let X be a maximal spanning set ; if it were dependent, say $x \in X$ is dependent on the rest of X, then we could omit x and still have spanning set , by (D2); this contradicts minimality, hence X is a basis .

Converse , if X is a basis ,it is independent, so no element is dependent on the rest , and X is a minimal spanning set.

3-Properties of groups with basic property

Definition(3-1) [1]: Let G be a group we said to be G a group with basic property if any two distinct bases for the subgroup H of G have the same number of elements and denoted by B-group .

Theorem(3-1) [1]:- The image of B-group by a group homomorphism is also form a B-group.

Proof.

Let G be B-group and let $|G|$ =n

We prove our theorem by induction on the order n of G.

If n=1 then the theorem is true.

Suppose n >1 , if H the image of G by the homomorphism

i.e $f:G\rightarrow H$ if f(G)=H

And $|H| = n$ then H≈ G

∴ H is B-group.

Next suppose $|H|<n$

Since H is a finite group

∴H has a basis

Suppose that B= $\{x_1, \ldots,x_m\}$. , C=$\{y_1, \ldots,y_k\}$. two basis to H ,we must prove k=m

We choose the elements $a_1, \ldots,a_m$, $b_1, \ldots,b_k$ of G such that :

$f(a_i) = x_i$; $1\leq i \geq m$;

$f(b_j) = y_j$; $1\leq j \geq k$

we choose the subgroup of G which generated by the elements $a_1, \ldots,a_m$ that is $K=< a_1, \ldots,a_m>$

If K≠ G

Then $f(K)= f(=< a_1, \ldots,a_m>) = < f(a_1), \ldots,f(a_m)>= < x_1, \ldots,x_m> \leq H$

Since G is B- group then $K=< a_1, \ldots,a_m>$ is B-group

By the induction hypothesis then H which its order less than the order of G is B-group

∴ H is B- group since it is the image of group homomorphism of B-group of order n.

And since H is B-group

∴$|B|$ =$|C|$ that is m=k

In the same way we can prove the case where $G\neq < b_1, \ldots,b_k>$

Finally study the following case $< a_1, \ldots,a_m>= G=< b_1, \ldots,b_k>$ we must prove B is basis to G that is we must prove no element in B is depend ent on the remaining members of B.

Suppose the conversely that is $a_1 \in< a_1, \ldots,a_m>$

Thus $x_1 =f(a_1) \in f(< a_1, \ldots,a_m>)= (<f(a_1), \ldots,f(a_m)>)= (< x_1, \ldots,x_m>)$ this contradicts $\{x_1, \ldots,x_m\}$.is basis to H.

∴$\{ a_1, \ldots,a_m\}$ is basis to G.

In the same way we can prove $\{ b_1, \ldots,b_k\}$ is basis to G

And since G is B- group thus m=k.

Definition(3-2): Let G be a group then we called the intersect of all maximal subgroups of G Frattini subgroup and denoted by Φ(G) ,either if there is no intersection then G=Φ(G).

Theorem (3-2): Every p-group is B-group where p is prime number.

Proof.

Let X=$\{x_1, \ldots,x_m\}$ is the basis to the group G and let$|X| = m$

We take the quotient group $G_o= G/\Phi(G)$ (Φ(G) is frattini subgroup of G).

And G_o is abelian group we consider it vector space on the field GF(p) which has p element

Let r be the dimension of vector space

Thus by Burnside theorem there exist a map transfers the set X to basis for the vector space G_o such that the deferent elements of X transfers to deferent elements of the basis of G_o this implies $|X| = r$

And m=r

And consequently the number of element of any basis to G is r.

If H be the subgroup of the P-group then H is also p-group consequently any two basis to the subgroup H of G are equal

∴G is B-group

Definition (3-3) [9]:-G is called periodic group if every element of G has a finite order .

Definition (3-3)[1]: A group G is said to be Quaziprimary group if the order of each element of G is power of a prime number p or q ;p≠q.

Theorem(3-3):-Every B-group is periodic .

Proof.

Let G be B-group to prove G is periodic group

Suppose that the conversely that is **a** ∊ G has infinite order

Thus $< a>=< a^2,a^3>$, $a^2\notin< a^3>$, $a^3\notin< a^2>$

∴ $\{a^2, a^3\}$ forms basis to the group G and also {a} this contradiction since is B-group.

Thus G is periodic.

Theorem (3-4): Every B-group is quasiprimary group.

Proof.

Let G be B-group to prove G is quasiprimary that every element of G is power of **P** or **q** .Let $|a| = mn$ where m,n>1 such that m, n prime number

Thus $< a>= < a^m> < a^n>= < a^m, a^n>$

Hence the set $\{ a^m, a^n\}$ is the smallest set generated $< a>$ this contradiction

∴the order of every element of G is prime

∴ G is quasiprimary group.

Corollary (3-1) LeT G be B-group and let a,b ∊ G such that the order of a is p^m and the order of b is q^n ,n,m ∊ N^+ then ab≠ba such that p, q is distinct prime number.

Proof.

Let ab =ba

∴ $|ab| = pnqn$

Thus ab has composite order in B-group this contradiction

∴ ab≠ba

Corollary (3-2) :Let G be a finite nilpotent B-group then G is p-group .

Proof.

Let G is finite nilpotent B-group

This by theorem of [Kurosh][5] every a finite nilpotent can be written as direct product of sylow subgroup :$G=G_1 \times G_2 \times \ldots \times G_m$

Such that G_i is p_i-group where p_i is prime number and $p_i \neq q_j$; $i \neq j$ $i=1,2,\ldots,m$.

If m> 1 this means there exist two commuting elements have order power of two distinct prime number .

This contradiction with the corollary (3-1)

∴ G is p-group .

Theorem(3-5): Every subgroup of B-group is also B-group.

Proof. Clearly .

Theorem(3-6): Every simple finite group with basic property (B-group) is a belian and its order is prime number.

Proof.

Let G be B-group and G be simple finite group .Suppose that G is not abelian

Since G is quasiprimary group and by theorem 16 in [8].G must be isomorphic with one of the following groups PSL(2,5) , PSL(2,7) , PSL(2,8) , PSL(2,17) , PSL(2,19) PSL(3,4) $S_z(8)$, $S_z(32)$. Such that PSL(n,q) (Linear special presentation groups over the field which has q elements) and $S_z(q)$ is Suzuki simple group over the field of q elements

Thus $S_z(q)$ is generated by only two elements [8]

Also PSL(n,q) have basis consist of only two elements and by fiet-Thompson [3] the order of nonabelian simple group must be even number thus the order of its elements is **2** thus its element is invertible

Let X be the set of all invertible elements of G , we must prove that G=<X>.

Since the conjugate element of invertible element is also invertible then the set X generated the normal subgroup of G and since G is simple thus G=<X>and X generated the group G .

∴It is contain basis as Y to the group G.

Since G is B-group thus it must be has basis consist of only two elements

∴$|Y|=2$ and the group G is generated by two invertible elements

This implies $G \cong D_4$ (D_4 dehadral group)but D_4 is not simple group this contradiction .

∴ G is abelian group and its order is prime number

Theorem(3-7) : Let G be a group with basic property then G is solvable .

Proof.

Suppose the converse .Let G be not solvable group thus G has simple and not abelian subgroup H

∴ H is B-group

Since the intersection of B-groups form B-group

And since there is no in G the intersection of simple not abelian group by theorem (3-5) thus G is solvable .

References

1-Al Khalaf A. Finite group with basically property,Dok. Acad. Nauk BSSR, N.11,1989.
2-Cohn P.M. Universal algebra, Harper & Row, New York, Evanston & London John Weather Hill. INC ,Tokyo, 1965.
3-Gross F. some remarks on groups admitting a fixed point free automorphism , Can. J. Mav. V.20 , 1968.
4-Jones P.,A basis Properties for invers semigroups,J Algebra ,V. 50.1978.
5-Kurosh A. ,Theory groups , NUKA , Moscow , 648p, 1967.
6-Lang S. Algebra, Columbia Uni, New York,1965
7-Shiryaev V. , Finite filtering Semigroup ,Acad .Nauk, Bssr, N.1,1993.
8-Suzuki M. Group theory II .New york ,1980.
9-Burnside W. Theory of groups of finite order, Cambridge Uni. , 1911.

دراسة في خواص الزمر المحققة لخاصية الأساس

دنيا علاوي جروان

E-mail :Dunia_alaja@yahoo.com

الخلاصة

الهدف من هذا البحث هو دراسة مفهوم الارتباط والاستقلال والأساس لبعض البنى الجبرية ثم اعطينا تعريف الزمرة المحققة لخاصية الأساس ودرسنا بعض خواصها الأساسية.