P- ISSN 1991-8941 E-ISSN 2706-6703 2024,(00), (00):00 – 00
Journal of University of Anbar for Pure Science (JUAPS) Open Access

A Survey on Riemannian Curvature Tensor for Certain Classes of Almost Contact Metric Manifolds

Mohammed Y. Abass

Department of Mathematics, College of Science, University of Basrah, Basra, Iraq;

ARTICLE INFO
Received: 05 / 12 /2023
Accepted: 13 / 01/ 2024
Available online: 07 / 02 / 2024

DOI: 10.37652/juaps.2024.145137.1167

Keywords:
Kenmotsu manifolds, $C(\lambda)$-manifolds, Sasakian manifolds, cosymplectic manifolds, Einstein manifolds.

ABSTRACT
This paper surveyed the components of Riemannian curvature tensor over the associated space of G-structure for certain classes of almost contact metric manifolds. These classes under consideration are only twelve and known as cosymplectic manifolds, Sasakian manifolds, Kenmotsu manifolds, C_9-manifolds, C_{12}-manifolds, normal manifolds of Killing type (CNK-manifold), nearly Kenmotsu manifolds, locally conformal almost cosymplectic manifolds (LCAC-manifolds), quasi-Sasakian manifolds, almost $C(\lambda)$-manifolds, nearly cosymplectic manifolds, and Kenmotsu type manifolds.

Introduction

The Riemannian curvature tensor (RC-tensor) is one of the interesting fields in the studying differential geometry. The Riemannian manifold of flat RC-tensor is locally isometric to the Euclidean space. Also, RC-tensor win its importance in the gravity theory and general relativity theory because its contraction is the Ricci tensor that a central mathematical tool in Einstein's theory. Based on the above, many authors studied RC-tensor of the manifolds and specially the almost contact metric manifolds that classified by D. Chinea and C. Gonzalez [1]. Especially among them, E. S. Volkova [2] determined the components of RC-tensor of CNK-manifolds. S. V. Umnova [3] established the components of RC-tensor of Kenmotsu manifolds and generalized Kenmotsu manifolds (nearly Kenmotsu manifolds). V. F. Kirichenko and A. R. Rustanov [4] deduced the components of RC-tensor of quasi-Sasakian manifolds. N. N. Dondukova [5].

*Corresponding: author at: Department of Mathematics, College of Science, University of Basrah, Basra, Iraq;
ORCID:https://orcid.org/0000-0003-1095-9963 ;Tel:+9647813334083
E-mail address: mohammed.abass@uobasrah.edu.iq

found the components of RC-tensor of cosymplectic manifolds and Sasakian manifolds. S. V. Kharitonova [6].

Concluded the components of RC-tensor of LCAC-manifolds. V. F. Kirichenko and E. V. Kusova [7] studied the components of RC-tensor of weakly cosymplectic manifolds (nearly cosymplectic manifolds).

So, according to the previous, we summarize these results in this paper and more than ones to have a survey about the RC-tensor of almost contact metric manifolds.

Preliminaries

Definition 1. [8] A topological space M is said to be a smooth manifold of dimension n and denoted by M^n, if M is T_2 −space, second countable, locally homeomorphic to $\mathbb{R}^n$, and has a smooth structure.

The symbol $\mathcal{X}(M)$ denotes to the module of whole vector fields on M^n.

Definition 2. [8] A bilinear map $g : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathbb{R}$ is said to be a metric tensor on M^n, if g is symmetric and positive definite.

Definition 3. [1] If a Riemannian manifold (M^{2n+1}, g) is provided by triple of a structure tensor (Φ, η, ξ), where η, ξ, Φ are tensors over M of types (1, 0), (0, 1), and (1, 1) respectively, such that $\forall\, Z_1, Z_2 \in \mathcal{X}(M)$, the following achieved:

$$\eta(\xi) = 1; \quad \eta \circ \Phi = 0; \quad \Phi(\xi) = 0; \quad id + \Phi^2 = \eta \otimes \xi;$$

$g(\Phi Z_1, \Phi Z_2) + \eta(Z_1)\eta(Z_2) = g(Z_1, Z_2)$, then it is known an almost contact metric (ACM-) manifold and denoted by $(M^{2n+1}, \xi, \eta, \Phi, g)$.

Definition 4. [8] A connection on a smooth manifold M is a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by $\nabla(Z_1, Z_2) = \nabla_{Z_1} Z_2$ and it attains the subsequent properties:

(1) $\boldsymbol{\nabla_{f_1 Z_1 + f_2 Z_2} Z_3 = f_1 \nabla_{Z_1} Z_3 + f_2 \nabla_{Z_2} Z_3}$;
(2) $\boldsymbol{\nabla_{Z_3}(f_1 Z_1 + f_2 Z_2) = f_1 \nabla_{Z_3} Z_1 + f_2 \nabla_{Z_3} Z_2 + Z_3(f_1) Z_1 + Z_3(f_2) Z_2}$,

for all $f_1, f_2 \in C^\infty(M)$ and $Z_1, Z_2, Z_3 \in \mathcal{X}(M)$.

Lemma 1. [8] Suppose that ∇ is a connection over M and $U, V \in \mathcal{X}(M)$. If $U = 0$, or $V = 0$ then $\nabla_U V = 0$.

Definition 5. [8] A Riemannian connection over the Riemannian manifold (M, g) is a connection ∇ on M that possess the following properties:

(i) $\boldsymbol{\nabla_{Z_1} Z_2 - \nabla_{Z_2} Z_1 = [Z_1, Z_2]}$, where $\boldsymbol{[Z_1, Z_2] = Z_1 \circ Z_2 - Z_2 \circ Z_1}$;
(ii) $\boldsymbol{Z_1\big(g(Z_2, Z_3)\big) = g\big(\nabla_{Z_1} Z_2, Z_3\big) + g\big(Z_2, \nabla_{Z_1} Z_3\big)}$,

for all $Z_1, Z_2, Z_3 \in \mathcal{X}(M)$.

There are several classes of ACM-manifolds $(M^{2n+1}, \xi, \eta, \Phi, g)$. We define some of these classes according to their Riemannian connection as the following:

Table 1. Some defining classes

Classes	Defining conditions
Cosymplectic [9]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 = 0}$
Nearly cosymplectic [10]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + \nabla_{Z_2}(\Phi) Z_1 = 0}$
Kenmotsu [11]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + g(Z_1, \Phi Z_2)\xi = -\eta(Z_2)\Phi Z_1}$
Sasakian [12]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + \eta(Z_2) Z_1 = g(Z_1, Z_2)\xi}$
C_9 [13]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 = \eta(Z_2)\nabla_{\Phi Z_1}\xi - g(\Phi Z_1, \nabla_{Z_2}\xi)\xi}$
C_{12} [14]	$\boldsymbol{-\eta(Z_1)\{\eta(Z_2)\Phi(\nabla_\xi \xi) + g(\nabla_\xi \xi, \Phi Z_2)\xi\} = \nabla_{Z_1}(\Phi) Z_2}$
CNK [2]	**Normal and** $\boldsymbol{\nabla_{Z_1}(\eta) Z_2 + \nabla_{Z_2}(\eta) Z_1 = 0}$
Nearly Kenmotsu [15]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + \nabla_{Z_2}(\Phi) Z_1 = -\eta(Z_2)\Phi Z_1 - \eta(Z_1)\Phi Z_2}$
Kenmotsu type [16]	$\boldsymbol{\nabla_{Z_1}(\Phi) Z_2 + \eta(Z_2)\Phi Z_1 = \nabla_{\Phi Z_1}(\Phi)\Phi Z_2}$

for all $Z_1, Z_2 \in \mathcal{X}(M)$, where ∇ refer to Riemannian connection. Moreover, an ACM-manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is called normal if $2N + \xi \otimes d\eta = 0$, where for all $\mathcal{U}, \mathcal{V} \in \mathcal{X}(M)$:

$$N(\mathcal{U}, \mathcal{V}) = \frac{1}{4}\big([\Phi\mathcal{U}, \Phi\mathcal{V}] + \Phi^2[\mathcal{U}, \mathcal{V}] - \Phi[\Phi\mathcal{U}, \mathcal{V}] - \Phi[\mathcal{U}, \Phi\mathcal{V}]\big),$$

is the Nijenhuis tensor of the structure tensor Φ (see [2]).

Definition 6. [6] An ACM-manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is bearing an almost cosymplectic manifold if $d\Omega = 0$ and $d\eta = 0$, where

$$\Omega(Z_1, Z_2) = g(Z_1, \Phi Z_2) \quad \text{and}$$
$$2d\eta(Z_1, Z_2) = \nabla_{Z_1}(\eta) Z_2 - \nabla_{Z_2}(\eta) Z_1;$$
$$3d\Omega(Z_1, Z_2, Z_3) = \nabla_{Z_1}(\Omega)(Z_2, Z_3) + \nabla_{Z_2}(\Omega)(Z_3, Z_1) + \nabla_{Z_3}(\Omega)(Z_1, Z_2), \quad \text{for all } Z_1, Z_2, Z_3 \in \mathcal{X}(M).$$

Definition 7. [6] An ACM-manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is bearing a LCAC-manifold if the ACM-manifold $(M^{2n+1}, \tilde{\xi}, \tilde{\eta}, \Phi, \tilde{g})$ is an almost cosymplectic manifold, where $\tilde{\xi} = exp(\alpha)\xi$; $\tilde{\eta} = exp(-\alpha)\eta$; $\tilde{g} = exp(-2\alpha)g$, and α is a smooth function.

Definition 8. [17] An ACM-manifold M^{2n+1} is known as quasi-Sasakian manifold if $d\Omega = 0$ and M is normal.

Definition 9. [8] An RC-tensor of type (3, 1) on a Riemannian manifold (N,g) is a tensor $R:\mathcal{X}(N)\times\mathcal{X}(N)\times\mathcal{X}(N)\to\mathcal{X}(N)$ that defined by $R(Z_1,Z_2)Z_3=\left([\nabla_{Z_1},\nabla_{Z_2}]-\nabla_{[Z_1,Z_2]}\right)Z_3$, for all $Z_1,Z_2,Z_3\in\mathcal{X}(N)$, where ∇ is Riemannian connection over N. Furthermore, the RC-tensor R of type (4, 0) is given by the formula $R(Z_1,Z_2,Z_3,Z_4)=g(R(Z_3,Z_4)Z_2,Z_1)$, with $Z_4\in\mathcal{X}(N)$.

Definition 10. [18] The associated space of G-structure for an ACM-manifold $(M^{2n+1},\xi,\eta,\Phi,g)$ is a set of all A-frame $(x;\Upsilon_0=\xi,\Upsilon_1,\ldots,\Upsilon_n,\Upsilon_{\hat{1}},\ldots,\Upsilon_{\hat{n}})$, where $x\in M$, $\Upsilon_a=\frac{1}{\sqrt{2}}\left(\chi_a-\sqrt{-1}\Phi(\chi_a)\right)$, $\Upsilon_{\hat{a}}=\frac{1}{\sqrt{2}}\left(\chi_a+\sqrt{-1}\Phi(\chi_a)\right)$, $a=1,2,\ldots,n$, $\hat{a}=a+n$ and $\{\chi_0=\xi,\chi_1,\ldots,\chi_n,\chi_{\hat{1}},\ldots,\chi_{\hat{n}}\}$ is a basis of $\mathcal{X}(M)$ which satisfies $g(\chi_p,\chi_q)=\delta_{pq}$, for all $p,q=0,1,\ldots,2n$.

Lemma 2. [19] Suppose that $(M^{2n+1},\xi,\eta,\Phi,g)$ is an ACM-manifold and R its RC-tensor of kind (4, 0) with components R_{pqrs} on the associated space of G-structure. Then the subsequent relations are satisfied:

(1) $\boldsymbol{R_{pqrs}=-R_{qprs}}$;

(2) $\boldsymbol{R_{pqrs}=-R_{pqsr}}$;

(3) $\boldsymbol{R_{pqrs}=R_{rspq}}$;

(4) $\boldsymbol{R_{pqrs}+R_{psqr}+R_{prsq}=0}$,

where $p,q,r,s=0,1,\ldots,2n$.

Definition 11. [20] An ACM-manifold $(M^{2n+1},\xi,\eta,\Phi,g)$ is said to be an almost $C(\lambda)$-manifold if its RC-tensor R fulfill the following identity:

$$\begin{aligned}&g(R(Z_3,Z_4)Z_2,Z_1)\\&=g(R(\Phi Z_3,\Phi Z_4)Z_2,Z_1)\\&-\lambda\{g(Z_1,Z_4)g(Z_2,Z_3)\\&-g(Z_1,Z_3)g(Z_2,Z_4)\\&-g(Z_1,\Phi Z_4)g(Z_2,\Phi Z_3)\\&+g(Z_1,\Phi Z_3)g(Z_2,\Phi Z_4)\},\end{aligned}$$

where $Z_1,Z_2,Z_3,Z_4\in\mathcal{X}(M)$, and $\lambda\in\mathbb{R}$. Moreover, a normal almost $C(\lambda)$-manifold is said to be $C(\lambda)$-manifold.

The Components of Riemannian Curvature Tensor on the Associated Space of G-Structure

In this section, we review the ingredients of RC-tensor on the associated space of G-structure for certain classes of ACM-manifolds.

Theorem 1. [5] The components of RC-tensor of cosymplectic manifolds are given by: $R_{\hat{a}bc\hat{d}}=A^{ad}_{bc}$, and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc}=A^{ad}_{[bc]}=0$.

Theorem 2. [5] The components of RC-tensor of Sasakian manifolds are given by:

1. $R_{\hat{a}bc\hat{d}}=A^{ad}_{bc}-2\delta^a_b\delta^d_c-\delta^a_c\delta^d_b$;
2. $R_{\hat{a}\hat{b}cd}=\delta^{ab}_{cd}=\delta^a_c\delta^b_d-\delta^a_d\delta^b_c$;
3. $R_{\hat{a}0b0}=\delta^a_b$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc}=A^{ad}_{[bc]}=0$.

Theorem 3. [5] The components of RC-tensor of Kenmotsu manifolds are given by:

1. $R_{\hat{a}bc\hat{d}}=A^{ad}_{bc}-\delta^a_c\delta^d_b$;
2. $R_{\hat{a}\hat{b}cd}=\delta^{ab}_{dc}=\delta^a_d\delta^b_c-\delta^a_c\delta^b_d$;
3. $R_{\hat{a}0b0}=-\delta^a_b$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc}=A^{ad}_{[bc]}=0$.

Theorem 4. [20] The components of RC-tensor of almost $C(\lambda)$-manifolds are given by:

1. $R_{\hat{a}\hat{b}cd}=\lambda\delta^{ab}_{cd}$;
2. $R_{\hat{a}0b0}=\lambda\delta^a_b$;
3. $R_{\hat{a}bc\hat{d}}-R_{\hat{a}cb\hat{d}}=-\lambda\delta^{ad}_{bc}$,

and the other components are vanish or given by Lemma 2, or their conjugates.

Theorem 5. [13] The components of RC-tensor of C_9-manifolds are given by:

1. $R_{0a\hat{b}0}=F_{ac}F^{cb}$;
2. $R_{0ab0}=-F_{ab0}$;
3. $R_{0ab\hat{c}}=-F_{ab}{}^{c}$;
4. $R_{\hat{a}bc\hat{d}}=A^{ad}_{bc}+F^{ad}F_{bc}$;
5. $R_{abcd}=-2F_{a[c}F_{|b|d]}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$, and F^{ab}, F_{ab}, F_{ab0}, ${F_{ab}}^{c}$ are components of Kirichenko's fifth structure tensor F (see [16]) and their covariant derivatives respectively.

Theorem 6. [14] The components of RC-tensor of C_{12}-manifolds are given by:

1. $C^a_b - C^a\, C_b = R_{\hat{a}0b0}$;
2. $C^{a\,b} - C^a\, C^b = R_{\hat{a}0\hat{b}0}$;
3. $A^{ad}_{bc} = R_{\hat{a}bc\hat{d}}$,

and the disappeared components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$, and C^a, C_a, $C^{a\,b}$, C^a_b are components of Kirichenko's sixth structure tensor G (see [16]) and their covariant derivatives respectively.

Theorem 7. [16] The components of RC-tensor over the manifolds of Kenmotsu type are seemed as follow:

1. $-\delta^a_c = R_{\hat{a}0c0}$;
2. $2A^a_{bcd} = R_{\hat{a}bcd}$;
3. $A^{ad}_{bc} - \delta^a_c\,\delta^d_b - {B^{ah}}_c\,{B_{bh}}^d = R_{\hat{a}bc\hat{d}}$;
4. $2\left(-\delta^a_{[c}\,\delta^b_{d]} + {B^{ab}}_{[cd]}\right) = R_{\hat{a}\hat{b}cd}$;
5. $-{B^{ab}}_h\,{B^{hd}}_c + {B^{ab}}_c{}^d = R_{\hat{a}\hat{b}c\hat{d}}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} and A^a_{bcd} are suitable smooth functions and ${B^{ab}}_c$, ${B_{ab}}^c$, ${B^{ab}}_{cd}$, ${B^{ab}}_c{}^d$ are components of Kirichenko's first structure tensor B (see [16]) and their covariant derivatives respectively.

Theorem 8. [6] The components of RC-tensor of LCAC-manifolds are appeared as follow:

1. $2\left(A^a_{bcd} - \alpha_0 B_{b[d}\delta^a_{c]} + 4\alpha^{[a}\delta^{h]}_{[c}B_{d]hb}\right) = R_{\hat{a}bcd}$;
2. $2\left(2\delta^{[b}_{[c}\,\alpha^{a]}_{d]} - \delta^a_{[c}\delta^b_{d]}\alpha_0^2 + 2B^{hab}B_{hdc}\right) = R_{\hat{a}\hat{b}cd}$;
3. $A^{ad}_{bc} - 4B^{dah}B_{chb} + 4\alpha^{[a}\delta^{h]}_c\alpha_{[h}\delta^d_{b]} - \delta^a_c\delta^d_b\alpha_0^2 + B^{ad}B_{bc} = R_{\hat{a}bc\hat{d}}$;
4. $2\left(2B_{[c|ab|d]} + B_{a[c}B_{d]b} - 2\alpha_{[a}B_{b]cd}\right) = R_{abcd}$;
5. $2\left(\alpha_{0[c}\delta^a_{d]} - 2\alpha^{[a}\delta^{h]}_{[c}B_{d]h} + B^{ab}B_{bcd}\right) = R_{\hat{a}0cd}$;
6. $A^{ac0}_b - \delta^c_b\alpha_0\alpha^a + \alpha_b B^{ac} = R_{\hat{a}b\hat{c}0}$;
7. $2B_{cab}\alpha_0 + 2B_{cab0} = R_{abc0}$;
8. $-\delta^a_b\alpha_{00} - B_{cb}B^{ac} - \delta^a_b\alpha_0^2 - \alpha^a\alpha_b - \alpha^a_b + 2\alpha^{[a}\delta^{c]}_b\alpha_c = R_{\hat{a}0b0}$;
9. $2\alpha_0 B^{ab} + 2B^{bac}\alpha_c - D^{ab0} - \alpha^{ab} - \alpha^a\alpha^b = R_{\hat{a}0\hat{b}0}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc}, A^{ac0}_b and A^a_{bcd} are suitable smooth functions, B^{abc}, B_{abc}, B_{abcd}, B_{abc0} are components of Kirichenko's second structure tensor C (see [16]) and their covariant derivatives respectively, B^{ab}, B_{ab}, D^{ab0} are the components of Kirichenko's third structure tensor D (see [21]) and their covariant derivatives respectively, α^a, α_a, α_0 are the components of $d\alpha$, α^a_b, α^{ab} are the components of $d\alpha^a$, and α_{00}, α_{0a} are the components of $d\alpha_0$.

Theorem 9. [4] The components of RC-tensor of quasi-Sasakian manifolds are given by:

1. $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - 2B^a_b B^d_c - B^a_c B^d_b$;
2. $R_{\hat{a}b0c} = B^a_{bc}$;
3. $R_{\hat{a}b0\hat{c}} = B^{ac}_b$;
4. $R_{\hat{a}0b0} = B^a_c B^c_b$;
5. $R_{\hat{a}\hat{b}cd} = 2B^a_{[c}B^b_{d]}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are smooth functions satisfy $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$, and B^a_b, B^{ac}_b, B^a_{bc}, are components of Kirichenko's fourth structure tensor E (see [16]) and their covariant derivatives respectively.

Theorem 10. [2] The components of RC-tensor of CNK-manifolds are given by:

1. $R_{\hat{a}bcd} = 2A^a_{bcd}$;
2. $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - 2{B_b}^a{B_c}^d - {B_c}^a{B_b}^d + {B^{ah}}_c{B_{hb}}^d$;
3. $R_{\hat{a}bc0} = -C^a_{bc} - B^{\ h}_{[b}{B_{c]h}}^a$;
4. $R_{\hat{a}0b0} = {B_b}^h{B_h}^a$;
5. $R_{\hat{a}\hat{b}cd} = 2\left({B^{ab}}_{[dc]} + B^{\ a}_{[c}B^{\ b}_{d]}\right)$;
6. $R_{\hat{a}\hat{b}c0} = 2{B_h}^{[a}{B^{b]h}}_c$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc}, A^a_{bcd} are suitable smooth functions, ${B_b}^a$, C^a_{bc} are the components of Kirichenko's fourth structure tensor E and their covariant derivatives respectively, and ${B^{ab}}_c$, ${B_{ab}}^c$, ${B^{ab}}_{cd}$ are the components of Kirichenko's first structure tensor B and their covariant derivatives respectively.

Theorem 11. [15] The components of RC-tensor of nearly Kenmotsu manifolds are given by:

1. $R_{\hat{a}bcd} = -\frac{2}{3}\delta^a_b F_{cd} + \frac{1}{3}\delta^a_c F_{db} + \frac{1}{3}\delta^a_d F_{bc}$;

2. $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - C^{adh}C_{hbc} - \frac{1}{2}F^{ad}F_{bc} - \delta^a_c\delta^d_b$;
3. $R_{\hat{a}\hat{b}cd} = 2C^{abh}C_{hcd} + F^{ab}F_{cd} - 2\delta^a_{[c}\delta^b_{d]}$;
4. $R_{\hat{a}\hat{b}\hat{c}\hat{d}} = C^{acdb} - \frac{1}{2}\left(F^{ab}F^{cd} + F^{ac}F^{db} + F^{ad}F^{bc}\right)$;
5. $R_{\hat{a}00b} = F^{ac}F_{cb} + \delta^a_b$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are suitable smooth functions, F^{ab}, F_{ab} are the components of Kirichenko's fifth structure tensor F, and C^{abc}, C_{abc}, C^{abcd} are the components of Kirichenko's second structure tensor C and their covariant derivatives respectively.

Theorem 12. [7] The components of RC-tensor of nearly cosymplectic manifolds are given by:

1. $R_{abcd} = -2B_{ab[cd]}$;
2. $R_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd}$;
3. $R_{\hat{a}0b0} = C^{ac}C_{bc}$;
4. $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}$,

and the other components are vanish or given by Lemma 2, or their conjugates, where A^{ad}_{bc} are suitable smooth functions, C^{ab}, C_{ab} are the components of Kirichenko's third structure tensor D, and B^{abc}, B_{abc}, B_{abcd} are the components of Kirichenko's second structure tensor C and their covariant derivatives respectively.

Conclusions

This paper collected the theories that determined the components of RC-tensors for 12 different classes of ACM-manifolds. So, the readers can be recognized the difference among these classes from the theorems in this paper. Then we concluded that the RC-tensor distinct according to its class.

Acknowledgments

The author expresses their sincere gratitude to the referees for their valuable comments, which made it possible to significantly improve this paper.

Conflict of Interest

The author declares no conflict of interest.

References

[1] Chinea, D., & Gonzalez, C. (1990). A classification of almost contact metric manifolds. *Annali di Matematica Pura ed Applicata*, 156(1), 15-36.

[2] Volkova, E. S. (1997). Curvature identities for normal manifolds of Killing type. *Mathematical Notes*, 62(3), 296-305.

[3] Umnova, S. V. (2002). Geometry of Kenmotsu manifolds and their generalizations. Ph. D. thesis, Moscow State Pedagogical University. (In Russian).

[4] Kirichenko, V. F., & Rustanov, A. R. (2002). Differential geometry of quasi-Sasakian manifolds. *Sbornik: Mathematics*, 193(8), 1173–1201.

[5] Dondukova, N. N. (2006). Geodesic transformations of almost contact metric manifolds. Ph. D. thesis, Moscow State Pedagogical University. (In Russian).

[6] Kharitonova, S. V. (2009). On the geometry of locally conformally almost cosymplectic manifolds. *Mathematical Notes*, 86(1-2), 121-131.

[7] Kirichenko, V. F., & Kusova, E. V. (2011). On geometry of weakly cosymplectic manifolds. *Journal of Mathematical Sciences*, 177(5), 668–674.

[8] Boothby, W. M. (1975). An introduction to differentiable manifolds and Riemannian geometry. Academic Press.

[9] Yoldaş, H. İ., Merİç, Ş. E., & Yaşar, E. (2021). Some special vector fields on a cosymplectic manifold admitting a Ricci soliton. *Miskolc Mathematical Notes*, 22(2), 1039-1050.

[10] Rustanov, A. (2022). Nearly cosymplectic manifolds of constant type. *Axioms*, 11(4), 152.

[11] Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds. *Tôhoku Mathematical Journal*, 24(1), 93-103.

[12] De Nicola, A., Dileo, G., & Yudin, I. (2018). On nearly Sasakian and nearly cosymplectic manifolds. *Annali di Matematica*, 197(1), 127-138.

[13] Rustanov, A. R., Yudin, A. I., & Melekhina, T. L. (2019). Geometry of strictly pseudo-cosymplectic manifolds. *Izvestiya Vuzov. Severo-Kavkazskii Region. Natural Science*, (1), 33-40. (In Russian).

[14] Abass, M. Y. (2020). Geometry of certain curvature tensors of almost contact metric manifold, Ph. D. thesis, University of Basrah. https://faculty.uobasrah.edu.iq/uploads/publications/1620305696.pdf

[15] Abu-Saleem, A., Kochetkov, I. D., & Rustanov, A. R. (2020, May). Two subclasses of generalized Kenmotsu manifolds. Materials Science and Engineering: Conference Series (Vol. 918, p. 012062). IOP Publishing.

[16] Abood, H. M., & Abass, M. Y. (2021). A study of new class of almost contact metric manifolds of Kenmotsu type. *Tamkang Journal of Mathematics*, 52(2), 253-266.

[17] Blair, D. E. (1967). The theory of quasi-Sasakian structures. *Journal of Differential Geometry*, 1(3-4), 331-345.

[18] Kirichenko, V. F. (2013). Differential geometric structures on manifolds. Odessa: Pechatnyy dom. (In Russian).

[19] Lee, J. M. (1997). Riemannian manifolds: An introduction to curvature. Springer-Verlag.

[20] Rustanov, A. R., Polkina, E. A., & Kharitonova, S. V. (2022). Projective invariants of almost $C(\lambda)$-manifolds. *Annals of Global Analysis and Geometry*, 61, 459-467.

[21] Kirichenko, V. F., & Dondukova, N. N. (2006). Contactly geodesic transformations of almost contact metric structures. Mathematical Notes, 80(2), pp. 204-213.

مراجعة حول تنسر الانحناء الريماني لبعض فئات المنطويات المترية التلامسية تقريباً

محمد يوسف عباس

قسم الرياضيات ، كلية العلوم ، جامعة البصرة ، البصرة ، العراق
Email: mohammed.abass@uobasrah.edu.iq

الخلاصة:

استعرض هذا البحث مركبات تنسر الانحناء الريماني على الفضاء المتعلق بالبنية – G لبعض فئات المنطويات المترية التلامسية تقريباً. الفئات التي تم دراستنا هي اثنتا عشرة فئة فقط والمعروفة بالاسماء منطويات كوسمبلكتك ومنطويات ساساكي ومنطويات كينموتسو ومنطويات – C_9 ومنطويات – C_{12} والمنطويات الطبيعية من النوع المعدوم ومنطويات كينموتسو التقريبي ومنطويات التحويل الكونفورمي المحلي للكوسمبلكتك تقريباً ومنطويات شبه ساساكي ومنطويات – $C(\lambda)$ تقريباً ومنطويات كوسمبلكتك التقريبي والمنطويات من نوع كينموتسو.

الكلمات المفتاحية : منطويات كينموتسو، منطويات – $C(\lambda)$ ، منطويات ساساكي، منطويات كوسمبلكتك، منطويات اينشتاين.