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Bornological Soft Sets

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ABSTRACT

The concept of bounded set within this set or space is given by many authors. A bornology is defined on soft set to solve the problems of boundedness for the soft set. Also, we construct soft base and soft subbase as a part of fundamental construction for bornological soft sets. Furthermore, It is a natural to study fundamental construction for bornological soft sets, such as soft subspace, product soft bornology and soft bornological isomorphism. Additionally, a family of bornological soft sets can be a partial ordered set by partial ordered relation and we prove that the intersection of bornological soft sets is bornological soft sets but the union of bornological soft sets is not necessary to be bornological soft sets. The left-right translation is soft bornological isomorphism and the product of bornological soft sets are bornological soft sets. Finally, generate soft bornological structure whose elements are soft unbounded sets.

1-INTRODUCTION

Previously, to solve the limitation or bounded problem for any set or space, the concept of bounded set within this set or space are given. An idea emerged since 1977, [1] to form a structure is called bornology β on a set X to solve the problem of limitation for a set X or any space in general way. In other words, if we have set X a collection bornology of subsets of X such that β covers X also, β is stable under hereditary and finite union see [2], [3], [4], [5]. Molodtsov [6] proposed the soft set theory in 1999 as a new mathematical tool for dealing with uncertainty modelling problems. See [7-14].

The main goal of this work is to solve the problems of boundedness for the soft set by constructing new structure that is called bornological soft sets. Also, we constructed new soft bornological structures in different ways. For example, we constructed soft bornology from soft subbase as well as, we give many results and properties on bornological soft sets. It is a natural to study a fundamental construction of this new structure such as subspace of soft bornological, product soft bornology and soft bornological isomorphism. Furthermore, that a family of bornological soft sets can be partial ordered set, every soft power set of a soft set is bornological soft sets, the composition of two soft bounded maps is soft bounded map, the intersection of bornological soft sets is soft bornological set, the left-right translation is soft bornological isomorphism and the product of bornological soft sets is bornological soft sets. Finally, generate soft bornological structure whose elements are soft unbounded sets.

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2-PRELIMINARIES

The definition and some results about the soft set theory are presented.

Definition(2-1) [6]**:** U denotes a universal set, while E denotes a set of parameters. The pair (ξ, A) is called a ***soft set*** under U where consisting of a subset A of E and a mapping $\xi: A \longrightarrow P(U)$.

i.e. $\xi = \{(e, \xi(e)), e \in A, \xi(e) \in P(U)\}$.

Propositions (2-2) [7]**:**

1. If $A = \emptyset$, then $\xi(e) = \emptyset$, we write $(\emptyset, \emptyset)$ is an empty soft set $\tilde{\emptyset}$.
2. If $A = E$, then $\xi(e) = U$, we write (U, E) is a universal soft set $\tilde{U}$.
3. If $L \subseteq A$, and $\xi(e) \subseteq \xi'(e)$, we write $(\xi, L) \tilde{\subseteq} (\xi', A)$.
4. If $L = A$, $\xi(e) \subseteq \xi'(e)$ and $\xi'(e) \subseteq \xi(e)$, we write $(\xi, L) = (\xi', A)$ to soft set are equal.
5. The soft union and soft intersection are.

$\xi(e) \cup \xi'(e), e \in L \cup A$, we write $(\xi, L) \tilde{\cup} (\xi', A)$
$\xi(e) \cap \xi'(e), e \in L \cap A$, we write $(\xi, L) \tilde{\cap} (\xi', A)$.

6. $(\xi, A)^c = \{\xi(e)^c : e \in E\}$, where $\xi(e)^c = U \backslash \xi(e)$.
7. The composition of two soft sets is soft set.

Example(2-3) [8]**:** Suppose that there are five cars in $U = \{c_1, c_2, c_3, c_4, c_5\}$ and
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is a set of parameters.
$e_1 =$ costly, $e_2 =$ beautiful, $e_3 =$sport, $e_4 =$ cheap, $e_5 =$ luxury,
$e_6 =$ modern, $e_7 =$ in good repair, $e_8 =$ in bad repair.
The parameters $e_i \in E$. $L = \{e_2, e_3, e_4, e_5, e_7\}$
$\xi(e_2) = \{c_2, c_3, c_5\}, \xi(e_3) = \{c_2, c_4\}, \xi(e_4) = \{c_1\}, \xi(e_5) = U, \xi(e_7) = \{c_3, c_5\}. A = \{e_1, e_2, \dots, e_7\}$
$\xi'(e_1) = \{c_3, c_5\}, \xi'(e_2) = \{c_4\}, \xi'(e_3) = \{c_2, c_4\}, \xi'(e_4) = \{c_1\}, \xi'(e_5) = \{c_2, c_3, c_4, c_5\}$,
$\xi'(e_6) = \xi'(e_7) = \{c_3\}. (\xi, L) \tilde{\cap} (\xi', A) = (\xi \cap \xi', \{e_3, e_4, e_5, e_7\})$, where $(\xi \cap \xi')(e_3) = \{c_2, c_4\}$,
$(\xi \cap \xi')(e_4) = \{c_1\}, (\xi \cap \xi')(e_5) = \{c_2, c_3, c_4, c_5\}, (\xi \cap \xi')(e_7) = \{c_3\}$. $(\xi, L) \tilde{\cup} (\xi', A) = (\xi \cup \xi', \{e_1, e_2, \dots, e_7\})$,
where $(\xi \cup \xi')(e_1) = \{c_3, c_5\}. (\xi \cup \xi')(e_2) = \{c_2, c_3, c_4, c_5\}, (\xi \cup \xi')(e_3) = \{c_2, c_4\}, (\xi \cup \xi')(e_4) = \{c_1\}, (\xi \cup \xi')(e_5) = U, (\xi \cup \xi')(e_6) = \{c_3\}, (\xi \cup \xi')(e_7) = \{c_3, c_5\}$.
$(\xi, L)^c = (\xi^c, \{e_1, e_2, e_3, e_4, e_6, e_7, e_8\})$, where $\xi^c(e_1) = U, \xi^c(e_2) = \{c_1, c_4\}$,
$\xi^c(e_3) = \{c_1, c_3, c_5\}, \xi^c(e_4) = \{c_2, c_3, c_4, c_5\}, \xi^c(e_6) = U, \xi^c(e_7) = \{c_1, c_2, c_4\}, \xi^c(e_8) = U$.
Note: We denote the soft set by $\mathbb{X} = (\xi, A)$.

Definition (2-4) [7]: Let $\xi \in S(U)$, where $S(U)$ referred to the set of all soft sets over U, the soft power set of ξ is characterized with
$P(\xi) = \{\xi_i \tilde{\subseteq} \xi : i \in I\}$,

$$|P(\xi)| = 2^{\sum_{e \in E} |\xi(e)|}$$

where $|\xi(e)|$ is cardinality of $\xi(e)$.

3-BORNOLOGICAL SOFT SET

In this section, we constructe a new structure known as a bornological soft set and discuss its definition as well as some results.

Definition (3-1): $\mathbb{X}$ be a soft set. A bornology $\tilde{\beta}$ on $\mathbb{X}$ is a collection of subset of $\mathbb{X}$

1 $-$ $\tilde{\beta}$ covers $\mathbb{X}$;
2- If $\tilde{B} \in \tilde{\beta}$, $\exists$ $\tilde{A} \tilde{\subseteq} \tilde{B}$, then $\tilde{A} \in \tilde{\beta}$;
3 $-$ $\tilde{\beta}$ is stable under finite soft union.

Then, $(\mathbb{X}, \tilde{\beta})$ is a ***bornological soft set***. And, its elements are called soft bounded sets (BSS). We can satisfy the first condition in different ways.

If the whole set $\mathbb{X}$ belong to the bornology or $\forall\, x \in \mathbb{X}, \{x\} \in \tilde{\beta}$ or or $\mathbb{X} = \cup_{\tilde{B} \tilde{\in} \tilde{\beta}} \tilde{B}$

Now we discuss the types of soft bornological sets:

- Let $\mathbb{X}$ be a soft set, $\tilde{\beta}_{dis}$ be the collection of all soft bounded subsets of $\mathbb{X}$, then $(\mathbb{X}, \tilde{\beta}_{dis})$ is a discrete bornological soft set.
- Let $\mathbb{X}$ be a soft set, $\tilde{\beta}_{fin}$ be the collection of a finite soft bounded subsets of $\mathbb{X}$, then $(\mathbb{X}, \tilde{\beta}_{fin})$ is a finite bornological soft set.
- Let $\mathbb{X}$ be a soft set, $\tilde{\beta}_u$ be the collection of all usual bounded subsets of $\mathbb{X}$, then $(\mathbb{X}, \tilde{\beta}_u)$ is a usual bornological soft set.

We can denote for bornological soft set by BSS.

Definition (3-2): If $(\mathbb{X}, \tilde{\beta})$ be a bornological soft set. Then a ***soft base*** $\tilde{\beta}_0$ is sub collection of soft bornology $\tilde{\beta}$, and each element of the soft bornology is contained in an element of the soft base.

Example (3-3): Suppose $U = \{r_1, r_2, r_3\}, A = \{e_1, e_2\}$, $\mathbb{X} = \{(e_1, \{r_1, r_2, r_3\}), (e_2, \{r_1, r_2, r_3\})\}$. And

Example (3-3): Suppose $U = \{r_1, r_2, r_3\}, A = \{e_1, e_2\}$,
$\mathbb{X} = \{(e_1, \{r_1, r_2, r_3\}), (e_2, \{r_1, r_2, r_3\})\}$. And
$\tilde{\beta} = \{\tilde{\emptyset}, \mathbb{X}, \{(e_1, \{r_1\})\}, \{(e_1, \{r_2\})\}, \{(e_1, \{r_3\})\}, \{(e_1, \{r_1, r_2\})\}, \{(e_1, \{r_1, r_3\})\}, \{(e_1, \{r_2, r_3\})\}, \{(e_2, \{r_1\})\}, \{(e_2, \{r_2\})\}, \{(e_2, \{r_3\})\}, \{(e_2, \{r_1, r_2\})\}, \{(e_2, \{r_1, r_3\})\}, \{(e_2, \{r_2, r_3\})\}, \{(e_1, \{r_1\}), (e_2, \{r_1\})\}, \{(e_1, \{r_1\}), (e_2, \{r_2\})\}, \{(e_1, \{r_1\}), (e_2, \{r_3\})\}, \{(e_1, \{r_1\}), (e_2, \{r_1, r_2\})\}, \{(e_1, \{r_1\}), (e_2, \{r_1, r_3\})\}, \{(e_1, \{r_1\}), (e_2, \{r_2, r_3\})\}, \{(e_1, \{r_2\}), (e_2, \{r_1\})\}, \{(e_1, \{r_2\}), (e_2, \{r_2\})\}, \{(e_1, \{r_2\}), (e_2, \{r_3\})\}$,

$\{(e_1,\{r_2\}),(e_2,\{r_1,r_2\})\},\{(e_1,\{r_2\}),(e_2,\{r_1,r_3\})\},$
$\{(e_1,\{r_2\}),(e_2,\{r_2,r_3\})\},\{(e_1,\{r_3\}),(e_2,\{r_1\})\},$
$\{(e_1,\{r_3\}),(e_2,\{r_2\})\},\{(e_1,\{r_3\}),(e_2,\{r_3\})\},$
$\{(e_1,\{r_3\}),(e_2,\{r_1,r_2\})\},\{(e_1,\{r_3\}),(e_2,\{r_1,r_3\})\},$
$\{(e_1,\{r_3\}),(e_2,\{r_2,r_3\})\},\{(e_1,\{r_1,r_2\}),(e_2,\{r_1\})\},$
$\{(e_1,\{r_1,r_2\}),(e_2,\{r_2\})\},\{(e_1,\{r_1,r_2\}),(e_2,\{r_3\})\},$
$\{(e_1,\{r_1,r_2\}),(e_2,\{r_1,r_2\})\},\{(e_1,\{r_1,r_2\}),(e_2,\{r_1,r_3\})\},$
$\{(e_1,\{r_1,r_2\}),(e_2,\{r_2,r_3\})\},\{(e_1,\{r_1,r_3\}),(e_2,\{r_1\})\},$
$\{(e_1,\{r_1,r_3\}),(e_2,\{r_2\})\},\{(e_1,\{r_1,r_3\}),(e_2,\{r_3\})\},$
$\{(e_1,\{r_1,r_3\}),(e_2,\{r_1,r_2\})\},\{(e_1,\{r_1,r_3\}),(e_2,\{r_1,r_3\})\},$
$\{(e_1,\{r_1,r_3\}),(e_2,\{r_2,r_3\})\},\{(e_1,\{r_2,r_3\}),(e_2,\{r_1\})\},$
$\{(e_1,\{r_2,r_3\}),(e_2,\{r_2\})\},\{(e_1,\{r_2,r_3\}),(e_2,\{r_3\})\},$
$\{(e_1,\{r_2,r_3\}),(e_2,\{r_1,r_2\})\},\{(e_1,\{r_2,r_3\}),(e_2,\{r_1,r_3\})\},$
$\{(e_1,\{r_2,r_3\}),(e_2,\{r_2,r_3\})\},\{(e_1,\{r_1\}),(e_2,\mathrm{U})\},$
$\{(e_1,\{r_2\}),(e_2,\mathrm{U})\},\{(e_1,\{r_3\}),(e_2,\mathrm{U})\},$
$\{(e_1,\{r_1,r_2\}),(e_2,\mathrm{U})\},\{(e_1,U)\},\{(e_2,U)\},$
$\{(e_1,\{r_1,r_3\}),(e_2,\mathrm{U})\},$
$\{(e_1,\{r_2,r_3\}),(e_2,\mathrm{U})\},\{(e_1,U),(e_2,\{r_1\})\},$
$\{(e_1,U),(e_2,\{r_2\})\},\{(e_1,U),(e_2,\{r_3\})\},$
$\{(e_1,U),(e_2,\{r_1,r_2\})\},$
$\{(e_1,U),(e_2,\{r_1,r_3\})\},\{(e_1,U),(e_2,\{r_2,r_3\})\}\}.$

It clear that $(\mathbb{X},\tilde{\beta})$ is a bornological soft set.
Now to find the soft base of $\tilde{\beta}$.

$\tilde{\beta}_0=\{\mathbb{X},\{(e_1,\{r_1,r_2\}),(e_2,\{r_1,r_2\})\},\{(e_1,\{r_1,r_2\}),$
$(e_2,\{r_1,r_3\})\},\{(e_1,\{r_1,r_2\}),(e_2,\{r_2,r_3\})\},$
$\{(e_1,\{r_1,r_3\}),(e_2,\{r_1,r_2\})\},\{(e_1,\{r_1,r_3\}),(e_2,\{r_1,r_3\})\},$
$\{(e_1,\{r_1,r_3\}),(e_2,\{r_2,r_3\})\},\{(e_1,\{r_2,r_3\}),(e_2,\{r_1,r_2\})\},$
$\{(e_1,\{r_2,r_3\}),(e_2,\{r_1,r_3\})\},\{(e_1,\{r_2,r_3\}),(e_2,\{r_2,r_3\})\},$
$\{(e_1,\{r_1\}),(e_2,\mathrm{U})\},\{(e_1,\{r_2\}),(e_2,\mathrm{U})\},$
$\{(e_1,\{r_3\}),(e_2,\mathrm{U})\},\{(e_1,\{r_1,r_2\}),(e_2,\mathrm{U})\},$
$\{(e_1,\{r_1,r_3\}),(e_2,\mathrm{U})\},$
$\{(e_1,\{r_2,r_3\}),(e_2,\mathrm{U})\},\{(e_1,U),(e_2,\{r_1\})\},$
$\{(e_1,U),(e_2,\{r_2\})\},\{(e_1,U),(e_2,\{r_3\})\},$
$\{(e_1,U),(e_2,\{r_1,r_2\})\},$
$\{(e_1,U),(e_2,\{r_1,r_3\})\},\{(e_1,U),(e_2,\{r_2,r_3\})\}\}.$
Or $\tilde{\beta}_0=\{\mathbb{X}\}$.

Definition (3-4): $(\mathbb{X},\tilde{\beta})$ is a bornological soft set, and $\mathbb{S}$ is a family of subset of $\mathbb{X}$, $\mathbb{S}$ is said to be soft subbase for $\tilde{\beta}$ if the family $\tilde{\beta}_0=\{\tilde{B}\,\tilde{\subseteq}\,\mathbb{X}:\mathbb{X}=\tilde{\cup}_{s\in\mathbb{S}}\ s, s\in\mathbb{S}\}$ forms a soft base for $\tilde{\beta}$.

Theorem (3-5): If $\mathbb{X}\neq\emptyset$, then a family $\mathbb{S}$ of subsets of $\mathbb{X}$, such that $\mathbb{X}=\tilde{\cup}_{s\in\mathbb{S}}\, s$ forms soft subbase for a soft bornology $\tilde{\beta}$ on $\mathbb{X}$.

Proof: To prove this, we must show that the family

$\tilde{\beta}_0=\{\tilde{B}\,\tilde{\subseteq}\,\mathbb{X}:\ \mathbb{X}=\tilde{\bigcup}_{s\in\mathbb{S}} s\ , s\in\mathbb{S}\}$ forms a soft base for a soft bornology on $\mathbb{X}$.

i. $\forall\, s\in\mathbb{S}\Longrightarrow s\,\tilde{\subset}\,\mathbb{X}\Longrightarrow s=s\,\tilde{\cup}\, s\Longrightarrow s\in\tilde{\beta}_0\Longrightarrow\mathbb{S}\,\tilde{\subseteq}\,\tilde{\beta}_0.$

Thus, $\mathbb{S}$ cover $\mathbb{X}$, then $\tilde{\beta}_0$ is covering $\mathbb{X}$.

ii. If $\tilde{B}_1,\tilde{B}_2\in\tilde{\beta}_0$

$$\tilde{\bigcup_{fin}} s_i \ \rightarrow \tilde{B}_1=\tilde{\cup}_{fin}\, s_i \qquad \tilde{\bigcup_{fin}} s_j\ ,\ \tilde{B}_2=\tilde{\cup}_{fin}\, s_j$$

$$\tilde{B}_1\,\tilde{\cup}\,\tilde{B}_2=\tilde{\cup}_{fin}\, s_k\ \tilde{\bigcup_{fin}} s_k\ \Longrightarrow\tilde{B}_1\,\tilde{\cup}\,\tilde{B}_2\in\tilde{\beta}_0.$$

That means there is $\tilde{\beta}$ on $\mathbb{X}$ for which $\tilde{\beta}_0$ form a soft base for $\tilde{\beta}$. Thus $\tilde{\beta}_0$ is forms soft base for a soft bornology on $\mathbb{X}$, by Definition of soft base, $\mathbb{S}$ form a soft subbase for $\tilde{\beta}$.

Example(3-6):
$U=\{2,4\},A=\{e_1,e_2\}.\mathbb{X}=\{(e_1,\{2,4\}),(e_2,\{2,4\})\},$
$\mathbb{S}=\{\{(e_1,\{2\})\},\{(e_1,\{4\})\},\{(e_2,\{2\})\},\{(e_2,\{4\})\}\}.$
Then $\mathbb{S}$ is a soft subbase for $\tilde{\beta}$ since
$\tilde{\beta}_0=\{\mathbb{X},\{(e_1,U)\},\{(e_2,U)\},\{(e_1,\{2\}),(e_2,U)\}$
$,\{(e_1,\{4\}),(e_2,U)\},$
$\{(e_1,U),(e_2,\{2\})\},\{(e_1,U),(e_2,\{4\})\}\}$. Or $\tilde{\beta}_0=\{\mathbb{X}\}$ are soft bases for $\tilde{\beta}$.

By the following theorem, we prove that a family of soft bornology can be partial ordered set by a partial ordering relation.

Theorem (3-7): Let $\{\tilde{\beta}_m\}_{m\in I}$ is refer to a collection of all bornological soft sets on $\mathbb{X}$, $''\leq''$ is a partial ordering relation if

$(\tilde{\beta}_m\leq\tilde{\beta}_n)\leftrightarrow(\forall\tilde{B}_m\in\tilde{\beta}_m\Rightarrow\forall\tilde{B}_m\in\tilde{\beta}_n)$ for all $m,n\in I$.

Proof:

i. Since $\tilde{\beta}_m\leq\tilde{\beta}_m$. Then $\tilde{\beta}_m\,\tilde{\subseteq}\,\tilde{\beta}_m$ for all $m\in I$. So, $''\leq''$ is reflective.
ii. Suppose that $\tilde{\beta}_m\leq\tilde{\beta}_n$ and $\tilde{\beta}_n\leq\tilde{\beta}_m$ for all $m,n\in I$. Then $\tilde{\beta}_m\,\tilde{\subseteq}\,\tilde{\beta}_n$ and $\tilde{\beta}_n\,\tilde{\subseteq}\,\tilde{\beta}_m$. So, $\tilde{\beta}_m=\tilde{\beta}_n$ and $''\leq''$ is anti-symmetric.
iii. Suppose that $\tilde{\beta}_m\leq\tilde{\beta}_n$ and $\tilde{\beta}_n\leq\tilde{\beta}_k$ for all $m,n,k\in I$. Then $\tilde{\beta}_m\,\tilde{\subseteq}\,\tilde{\beta}_n$ and $\tilde{\beta}_n\,\tilde{\subseteq}\,\tilde{\beta}_k$. So, $\tilde{\beta}_m\,\tilde{\subseteq}\,\tilde{\beta}_k$. It implies that $\tilde{\beta}_m\leq\tilde{\beta}_k$ and $''\leq''$ has transitive property.

Then $\{\tilde{\beta}_m\}_{m\in I}$ is partial order set.

Theorem (3-8): Every soft power set $P(\tilde{B})$ of soft set $\tilde{B}$ is bornological soft set over the soft set $\tilde{B}$.
Proof: Since the soft power set of $\tilde{B}$ is $P(\tilde{B})$.

To prove its bornological soft set , we must satisfy the three conditions:

i. Since $\tilde{B} \in P(\tilde{B})$, then $P(\tilde{B})$ covers $\tilde{B}$;

ii. If $\tilde{A} \tilde{\subseteq} \tilde{B}$ and $\tilde{B} \in P(\tilde{B})$, then $\tilde{A} \in P(\tilde{B})$; (hereditary property)

iii. Let $\tilde{A}_i \in P(\tilde{B})$, for all $i = 1, 2, \dots, n$ and since (union of finite soft set is also finite soft set)

$\tilde{\cup}_{1 \le i \le n} \tilde{A}_i$ is soft set. And $\tilde{\cup}_{1 \le i \le n} \tilde{A}_i \in P(\tilde{B})$. It follows that $\tilde{\cup}_{1 \le i \le n} \tilde{A}_i \in P(\tilde{B})$.

Then $P(\tilde{B})$ is soft bornology on $\tilde{B}$. Hence $\left(\tilde{B}, P(\tilde{B})\right)$ is called a bornological soft set.

Definition(3-9): Let $(\mathbb{X}, \tilde{\beta}_1), (\mathbb{Y}, \tilde{\beta}_2)$ be two bornological soft sets and $\psi: (\mathbb{X}, \tilde{\beta}_1) \longrightarrow (\mathbb{Y}, \tilde{\beta}_2)$. If the image for every soft bounded set in $\tilde{\beta}_1$ is soft bounded set in $\tilde{\beta}_2$, then ψ is called a soft bounded map.

That means $\forall\ \tilde{B} \in \tilde{\beta}_1 \longrightarrow \psi(\tilde{B}) \in \tilde{\beta}_2$.

Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ be three bornological soft sets and $\psi: \mathbb{X} \longrightarrow \mathbb{Y}$, $\phi: \mathbb{Y} \longrightarrow \mathbb{Z}$ be two soft bounded maps. From the definition of the composite map $\phi \circ \psi: \mathbb{X} \longrightarrow \mathbb{Z}$ is soft bounded map.

Proposition(3-10): The intersection of bornological soft sets is also bornological soft set $(\mathbb{X}, \tilde{\beta}_1 \tilde{\cap} \tilde{\beta}_2)$.

Proof:

i. Since $\tilde{\beta}$ and $\tilde{\beta}'$ are soft bornologies on $\mathbb{X}$ that mean the condition of the covering is hold. $\forall x \in \mathbb{X}$, $\{x\} \in \tilde{\beta}$ and $\forall x \in \mathbb{X}, \{x\} \in \tilde{\beta}'$. That means $\forall x \in \mathbb{X}, \{x\} \in \tilde{\beta}, \tilde{\beta}'$, then $\{x\} \in \tilde{\beta} \tilde{\cap} \tilde{\beta}'$. Then $\tilde{\beta} \tilde{\cap} \tilde{\beta}'$ cover $\mathbb{X}$.

ii. Let $\tilde{B} \in \tilde{\beta}_1 \tilde{\cap} \tilde{\beta}_2$. Then, $\tilde{B} \in \tilde{\beta}_1, \tilde{B} \in \tilde{\beta}_2$ Since $\tilde{\beta}_1, \tilde{\beta}_2$ soft bornologies , then if there exact $\tilde{M} \tilde{\subseteq} \tilde{B}$ *such that* $\tilde{M} \in \tilde{\beta}_1, \tilde{M} \in \tilde{\beta}_2$ thus $\tilde{M} \in \tilde{\beta}_1 \tilde{\cap} \tilde{\beta}_2$. S, the hereditary properties are hold.

iii. Let $\tilde{A}, \tilde{B} \in \tilde{\beta}_1 \tilde{\cap} \tilde{\beta}_2$. Then $\tilde{A}, \tilde{B} \in \tilde{\beta}_1$ and $\tilde{A}, \tilde{B} \in \tilde{\beta}_2$. Since $\tilde{\beta}_1, \tilde{\beta}_2$ soft bornologies , then $\tilde{A}\ \tilde{\cup}\ \tilde{B} \in \tilde{\beta}_1$ and $\tilde{A}\ \tilde{\cup}\ \tilde{B} \in \tilde{\beta}_2$. So, $\tilde{A}\ \tilde{\cup} \tilde{B}\ \in \tilde{\beta}_1 \tilde{\cap} \tilde{\beta}_2$.

Notice that $\tilde{\beta}_1 \tilde{\cap} \tilde{\beta}_2$ defines a soft bornology and $(\mathbb{X}, \tilde{\beta}_1 \tilde{\cap} \tilde{\beta}_2)$ is bornological soft set.

By the next example, we show that the union of two soft bornological sets is not necessarily to be a soft bornological set.

However the union of bornological soft sets $(\mathbb{X}, \tilde{\beta}_1 \tilde{\cup}\ \tilde{\beta}_2)$ is not necessary to be bornological soft set.

Example (3-11):Let $U_1 = \{R, Y\}, U_2 = \{R, B\}, A = \{e_1, e_2\}, \mathbb{X}_1 = \{(e_1, \{R, Y\}), (e_2, \{R, Y\})\}$,
$\mathbb{X}_2 = \{(e_1, \{R, B\}), (e_2, \{R, B\})\}$.

$\tilde{\beta}_1 = \{\tilde{\emptyset}, \mathbb{X}_1, \{(e_1, \{R\})\}, \{(e_1, \{Y\})\}, \{(e_2, \{R\})\}, \{(e_2, \{Y\})\}, \{(e_1, \{R\}), (e_2, \{R\})\}, \{(e_1, \{R\}), (e_2, \{Y\})\}, \{(e_1, \{R\}), (e_2, U_1)\}, \{(e_1, \{Y\}), (e_2, U_1)\}, \{(e_1, \{Y\}), (e_2, \{R\})\}, \{(e_1, \{Y\}), (e_2, \{Y\})\}, \{(e_1, U_1), (e_2, \{R\})\}, \{(e_1, U_1), (e_2, \{Y\})\}, \{(e_1, U_1)\}, \{(e_2, U_1)\}\}$.

$\tilde{\beta}_2 = \{\tilde{\emptyset}, \mathbb{X}_2, \{(e_1, \{R\})\}, \{(e_1, \{B\})\}, \{(e_2, \{R\})\}, \{(e_2, \{B\})\}, \{(e_1, \{R\}), (e_2, \{R\})\}, \{(e_1, \{R\}), (e_2, \{B\})\}, \{(e_1, \{R\}), (e_2, U_2)\}, \{(e_1, \{B\}), (e_2, \{R\})\}, \{(e_1, \{B\}), (e_2, \{B\})\}, \{(e_1, \{B\}), (e_2, U_2)\}, \{(e_1, U_2), (e_2, \{R\})\}, \{(e_1, U_2), (e_2, \{B\})\}, \{(e_1, U_2)\}, \{(e_2, U_2)\}\}$.

The union of soft bornologies $\tilde{\beta}_1$ and $\tilde{\beta}_2$

$U = \{R, Y, B\}, \mathbb{X} = \{(e_1, \{R, Y, B\}), (e_2, \{R, Y, B\})\}$.

$\tilde{\beta}_X = \{\tilde{\emptyset}, \mathbb{X}, \{(e_1, \{\mathrm{R}\})\}, \{(e_1, \{\mathrm{Y}\})\}, \{(e_1, \{B\})\}, \{(e_1, \{\mathrm{R, Y}\})\}, \{(e_1, \{\mathrm{R, B}\})\}, \{(e_1, \{\mathrm{Y, B}\})\}, \{(e_2, \{\mathrm{R}\})\}, \{(e_2, \{Y\})\}, \{(e_2, \{\mathrm{B}\})\}, \{(e_2, \{\mathrm{R, Y}\})\}, \{(e_2, \{\mathrm{R, B}\})\}, \{(e_2, \{\mathrm{Y, B}\})\}, \{(e_1, \{\mathrm{R}\}), (e_2, \{\mathrm{R}\})\}, \{(e_1, \{R\}), (e_2, \{Y\})\}, \{(e_1, \{\mathrm{R}\}), (e_2, \mathrm{B})\}, \{(e_1, \{\mathrm{R}\}), (e_2, \{\mathrm{R, Y}\})\}, \{(e_1, \{\mathrm{R}\}), (e_2, \{\mathrm{R, B}\})\}, \{(e_1, \{\mathrm{R}\}), (e_2, \{\mathrm{Y, B}\})\}, \{(e_1, \{\mathrm{Y}\}), (e_2, \{R\})\}, \{(e_1, \{\mathrm{Y}\}), (e_2, \{\mathrm{Y}\})\}, \{(e_1, \{\mathrm{Y}\}), (e_2, \{B\})\}, \{(e_1, \mathrm{Y}), (e_2, \{\mathrm{R, Y}\})\}, \{(e_1, \{Y\}), (e_2, \{\mathrm{R, B}\})\}, \{(e_1, \{\mathrm{Y}\}), (e_2, \{\mathrm{Y, B}\})\}, \{(e_1, \{\mathrm{B}\}), (e_2, \{\mathrm{R}\})\}, \{(e_1, \{\mathrm{B}\}), (e_2, \{\mathrm{Y}\})\}, \{(e_1, \{\mathrm{B}\}), (e_2, \{B\})\}, \{(e_1, \{\mathrm{B}\}), (e_2, \{\mathrm{R, Y}\})\}, \{(e_1, \{B\}), (e_2, \{\mathrm{R, B}\})\}, \{(e_1, \{B\}), (e_2, \{\mathrm{Y, B}\})\}, \{(e_1, \{\mathrm{R, Y}\}), (e_2, \{R\})\}, \{(e_1, \{\mathrm{R, Y}\}), (e_2, \{Y\})\}, \{(e_1, \{\mathrm{R, Y}\}), (e_2, \{\mathrm{B}\})\}, \{(e_1, \{\mathrm{R, Y}\}), (e_2, \{\mathrm{R, Y}\})\}, \{(e_1, \{\mathrm{R, Y}\}), (e_2, \{\mathrm{R, B}\})\}, \{(e_1, \{\mathrm{R, Y}\}), (e_2, \{\mathrm{Y, B}\})\}, \{(e_1, \{\mathrm{R, B}\}), (e_2, \{R\})\}, \{(e_1, \{\mathrm{R, B}\}), (e_2, \{\mathrm{Y}\})\}, \{(e_1, \{\mathrm{R, B}\}), (e_2, \{\mathrm{B}\})\}, \{(e_1, \{\mathrm{R, B}\}), (e_2, \{\mathrm{R, Y}\})\}, \{(e_1, \{\mathrm{R, B}\}), (e_2, \{\mathrm{R, B}\})\}, \{(e_1, \{\mathrm{R, B}\}), (e_2, \{\mathrm{Y, B}\})\}, \{(e_1, \{\mathrm{Y, B}\}), (e_2, \{\mathrm{R}\})\}, \{(e_1, \{\mathrm{Y, B}\}), (e_2, \{\mathrm{Y}\})\}, \{(e_1, \{\mathrm{Y, B}\}), (e_2, \{\mathrm{B}\})\}, \{(e_1, \{\mathrm{Y, B}\}), (e_2, \{\mathrm{R, Y}\})\}, \{(e_1, \{\mathrm{Y, B}\}), (e_2, \{\mathrm{R, B}\})\}, \{(e_1, \{\mathrm{Y, B}\}), (e_2, \{\mathrm{Y, B}\})\}, \{(e_1, \{\mathrm{R}\}), (e_2, \mathrm{U})\}, \{(e_1, \{\mathrm{Y}\}), (e_2, \mathrm{U})\}, \{(e_1, \{\mathrm{B}\}), (e_2, \mathrm{U})\}, \{(e_1, \{\mathrm{R, Y}\}), (e_2, \mathrm{U})\}, \{(e_1, \{\mathrm{R, B}\}), (e_2, \mathrm{U})\},$

$\{(e_1,\{\text{Y},\text{B}\}),(e_2,\text{U})\}$,
$\{(e_1,U),(e_2,\{R\})\},\{(e_1,U),(e_2,\{Y\})\}$,
$\{(e_1,U),(e_2,\{B\})\},\{(e_1,U),(e_2,\{R,Y\})\}$,
$\{(e_1,U),(e_2,\{R,B\})\}$,
$\{(e_1,U),(e_2,\{Y,B\})\},\{(e_1,U)\},\{(e_2,U)\}\}$.

It is clear that $\{(e_1,\{\text{Y},\text{B}\})\}$ do not belong to $\tilde{\beta}_1$ or $\tilde{\beta}_2$ this contradiction.

Definition (3-12): Let $\{\tilde{\beta}_m\}_{m\in I}$ a family of all soft bornological sets on $\mathbb{X}$. If $\tilde{\beta}_m \leq \tilde{\beta}_n$ then $\tilde{\beta}_n$ is soft finer than $\tilde{\beta}_m$. In this case $\tilde{\beta}_m$ is said to be soft coarser than $\tilde{\beta}_n$.

Proposition (3-13): Let $\psi: (\mathbb{X},\tilde{\beta}_1) \to (\mathbb{Y},\tilde{\beta}_2)$ and $\phi: (\mathbb{Y},\tilde{\beta}_2) \to (\mathbb{Z},\tilde{\beta}_3)$ be two soft bounded maps. Then the composition $\phi \circ \psi\ : (\mathbb{X},\tilde{\beta}_1) \to (\mathbb{Z},\tilde{\beta}_3)$ is soft bounded map.

Proof: Suppose $\tilde{B} \in \tilde{\beta}_1$ since $\psi: (\mathbb{X},\tilde{\beta}_1) \to (\mathbb{Y},\tilde{\beta}_2)$ is soft bounded map then $\psi(\tilde{B})$ is soft bounded set then $\psi(\tilde{B}) \in \tilde{\beta}_2$.

Again since $\phi: (\mathbb{Y},\tilde{\beta}_2) \to (\mathbb{Z},\tilde{\beta}_3)$ is soft bounded map. It follows that $\phi(\psi(\tilde{B})) \in \tilde{\beta}_3$.

So, the composition $\phi \circ \psi: (\mathbb{X},\tilde{\beta}_1) \to (\mathbb{Z},\tilde{\beta}_3)$ is soft bounded map.

Definition (3-14): Suppose $(\mathbb{X},\tilde{\beta})$ a bornological soft set and $\mathbb{G} \,\tilde{\subseteq}\, \mathbb{X}$. The $\tilde{\beta}_{\mathbb{G}} = \{\tilde{B} \,\tilde{\cap}\, \mathbb{G}: \tilde{B} \in \tilde{\beta}\}$ on $\mathbb{G}$ is called a relative soft bornology generated by the set $\mathbb{G}$.

Theorem (3-15): Let $(\mathbb{X},\tilde{\beta})$ be bornological soft set and $\mathbb{G} \,\tilde{\subseteq}\, \mathbb{X}$. Then the collection $\tilde{\beta}_{\mathbb{G}} = \{\tilde{B} \,\tilde{\cap}\, \mathbb{G}, \tilde{B} \in \tilde{\beta}\}$ is a soft bornology on $\mathbb{G}$..

Proof: To show that $\tilde{\beta}_{\mathbb{G}} = \{\tilde{B} \,\tilde{\cap}\, \mathbb{G}, \tilde{B} \in \tilde{\beta}\}$ is a soft bornology on $\mathbb{G}$. We must prove that:

i. Let $\tilde{A} = \tilde{B} \,\tilde{\cap}\, \mathbb{G}$ and $\tilde{B} \in \tilde{\beta}$, to prove that $\tilde{\beta}_{\mathbb{G}}$ is covering of $\mathbb{G}$, i.e. $\mathbb{G} = \cup_{\tilde{A}\in\tilde{\beta}_{\mathbb{G}}} \tilde{A}$

$$\tilde{\bigcup_{\tilde{A}\in\tilde{\beta}_{\mathbb{G}}}} \tilde{A} = \tilde{\bigcup_{\tilde{B}\in\tilde{\beta}}} (\tilde{B} \,\tilde{\cap}\, \mathbb{G}) = \tilde{\bigcup_{\tilde{B}\in\tilde{\beta}}} \tilde{B}) \,\tilde{\cap}\, \mathbb{G}$$

Where $\cup_{\tilde{B}\tilde{\in}\tilde{\beta}} \tilde{B} = \mathbb{X}$ and $\mathbb{X} \,\tilde{\cap}\, \mathbb{G} = \mathbb{G}$. Then $\mathbb{G} = \cup_{\tilde{A}\tilde{\in}\tilde{\beta}_{\mathbb{G}}} \tilde{A}$ then $\tilde{\beta}_{\mathbb{G}}$ is covering of $\mathbb{G}$.

ii. Let $\tilde{A} \in \tilde{\beta}_{\mathbb{G}}$, i.e. $\tilde{A} = \tilde{B} \,\tilde{\cap}\, \mathbb{G}$ and $\tilde{M} \,\tilde{\subseteq}\, \mathbb{G}, \tilde{M} \,\tilde{\subseteq}\, \tilde{A}\ thus\ \tilde{M} \,\tilde{\subseteq}\, \tilde{B} \,\tilde{\cap}\, \mathbb{G}$.

To prove $\tilde{M} \in \tilde{\beta}_{\mathbb{G}}$ we must satisfy that $\tilde{M} = \tilde{U} \,\tilde{\cap}\, \mathbb{G}$ where $\tilde{U} \in \tilde{\beta}$.

$\because \tilde{M} \,\tilde{\subseteq}\, \tilde{B} \,\tilde{\cap}\, \mathbb{G} \to \tilde{M} \,\tilde{\subseteq}\, \tilde{B} \,\tilde{\cap}\, \tilde{M} \,\tilde{\subseteq}\, \mathbb{G}$. $\therefore \tilde{B} \in \tilde{\beta}$, then $\tilde{M} \in \tilde{\beta}$. Take $\tilde{M} = \tilde{U}$,

$\therefore\ \tilde{M} = \tilde{U} \,\tilde{\cap}\, \mathbb{G}$ and $\tilde{U} \in \tilde{\beta}$. Then $\tilde{M} \in \tilde{\beta}_{\mathbb{G}}$.

iii. Let $\{\ \tilde{A}_i\ : i \in I, I\ finite\}$, be a finite soft element of $\tilde{\beta}_{\mathbb{G}}$, to prove $\tilde{\beta}_{\mathbb{G}}$ is stable under finite soft union. Since $\forall\, i \in I,\ \exists\, \tilde{B}_i \in \tilde{\beta}$ such that $\tilde{A}_i = \tilde{B}_i \,\tilde{\cap}\, \mathbb{G}$

$$\tilde{\bigcup_{i\in I}} \tilde{A}_i = \tilde{\bigcup_{i\in I}} (\tilde{B}_i \,\tilde{\cap}\, \mathbb{G}) \left(\tilde{\bigcup_{i\in I}} \tilde{B}_i\right) \tilde{\cap}\, \mathbb{G} = \tilde{B} \,\tilde{\cap}\, \mathbb{G}, i \in I, I\ finite \bigcup_{\tilde{A}\in\tilde{\beta}_{\mathbb{G}}} \tilde{A}$$

(since $\tilde{\beta}$ is stable under finite soft union). Then, $\tilde{B} \,\tilde{\cap}\, \mathbb{G} \in \tilde{\beta}_{\mathbb{G}}$ i.e. $\cup_{\tilde{i\in I}} \tilde{A}_i \in \tilde{\beta}_{\mathbb{G}}, I\ finite$

Theorem (3-16): Let $(\mathbb{X},\tilde{\beta})$ is a bornological soft set and $(\mathbb{G},\tilde{\beta}_{\mathbb{G}})$ be a subspace of bornological soft set $(\mathbb{X},\tilde{\beta})$. $\mathbb{H},\mathbb{G} \in \tilde{\beta}$ if $\mathbb{H} \,\tilde{\subseteq}\, \mathbb{G}$, then $\mathbb{H} \in \tilde{\beta}_{\mathbb{G}}$.

Proof: Suppose that $\mathbb{H} \in \tilde{\beta}$.Since $\mathbb{H} \,\tilde{\subseteq}\, \mathbb{G}, \mathbb{H} = \mathbb{H} \,\tilde{\cap}\, \mathbb{G}$ then $\mathbb{H} \in \tilde{\beta}_{\mathbb{G}}$ by assumption $\mathbb{H} \,\tilde{\subseteq}\, \mathbb{G}$ and $\mathbb{H} \in \tilde{\beta}$.

Theorem (3-17): If $(\mathbb{G},\tilde{\beta}_{\mathbb{G}})$ soft subspace of soft bornological set $(\mathbb{X},\tilde{\beta})$. Then the following are equivalent:

i. $\mathbb{G} \in \tilde{\beta}$.
ii. $\tilde{\beta}_{\mathbb{G}} \,\tilde{\subseteq}\, \tilde{\beta}$.

Proof: $(i) \longrightarrow (ii)$ Let $\mathbb{G} \in \tilde{\beta}$ and take as given $\mathbb{H} \in \tilde{\beta}_{\mathbb{G}}$. From the definition of bornological soft subspace, $\mathbb{H} = \tilde{m} \,\tilde{\cap}\, \mathbb{G}$, where $\exists\, \tilde{m} \in \tilde{\beta}$. Since $\mathbb{G} \in \tilde{\beta}, \tilde{m} \in \tilde{\beta}$ then, $\mathbb{H} \in \tilde{\beta}$.

$(ii) \longrightarrow (i)$ Assume that $\tilde{\beta}_{\mathbb{G}} \,\tilde{\subseteq}\, \tilde{\beta}$. Since $\mathbb{G} \in \tilde{\beta}_{\mathbb{G}}$ then, $\mathbb{G} \in \tilde{\beta}$.

Definition (3-18): Let $\mathbb{H} \,\tilde{\subseteq}\, \mathbb{G} \,\tilde{\subseteq}\, \mathbb{X}$, $(\mathbb{G},\tilde{\beta}_{\mathbb{G}})$ and $(\mathbb{H},\tilde{\beta}_{\mathbb{H}})$ are soft subspaces of bornological soft set of $(\mathbb{X},\tilde{\beta})$ and $\left(\mathbb{H},(\tilde{\beta}_{\mathbb{G}})_{\mathbb{H}}\right) \,\tilde{\subseteq}\, (\mathbb{G},\tilde{\beta}_{\mathbb{G}})$. Then, $\left(\mathbb{H},(\tilde{\beta}_{\mathbb{G}})_{\mathbb{H}}\right)$ is called bornological soft subspace of a bornological soft subspace $(\mathbb{G},\tilde{\beta}_{\mathbb{G}})$.

Theorem (3-19): Let $\mathbb{H} \,\tilde{\subseteq}\, \mathbb{G} \,\tilde{\subseteq}\, \mathbb{X}, (\mathbb{G},\tilde{\beta}_{\mathbb{G}})$ and $(\mathbb{H},\tilde{\beta}_{\mathbb{H}})$ are soft subspaces of bornological soft set $(\mathbb{X},\tilde{\beta})$, and $(\mathbb{H},(\tilde{\beta}_{\mathbb{G}})_{\mathbb{H}}) \,\tilde{\subseteq}\, (\mathbb{G},\tilde{\beta}_{\mathbb{G}})$.

Then, $\tilde{\beta}_{\mathbb{H}} = (\tilde{\beta}_{\mathbb{G}})_{\mathbb{H}}$.

Proof: $\tilde{\beta}_{\mathbb{H}} = \{\tilde{B} \,\tilde{\cap}\, \mathbb{H}: \tilde{B} \in \tilde{\beta}\}$ and $\tilde{\beta}_{\mathbb{G}} = \{\tilde{B} \,\tilde{\cap}\, \mathbb{G}: \tilde{B} \in \tilde{\beta}\}$.

Therefore :

$(\tilde{\beta}_{\mathbb{G}})_{\mathbb{H}} = \{(\tilde{B} \,\tilde{\cap}\, \mathbb{G}) \,\tilde{\cap}\, \mathbb{H}: \tilde{B} \in \tilde{\beta}\}$
$= \{\tilde{B} \,\tilde{\cap}\, (\mathbb{G} \,\tilde{\cap}\, \mathbb{H}): \tilde{B} \in \tilde{\beta}\}$
$= \{\tilde{B} \,\tilde{\cap}\, \mathbb{H}: \tilde{B} \in \tilde{\beta}\}$ since $(\mathbb{H} \,\tilde{\subseteq}\, \mathbb{G})\ = \tilde{\beta}_{\mathbb{H}}$.

Proposition (3-20): If $(\mathbb{F},\tilde{\beta}_{\mathbb{F}})$ and $(\mathbb{G},\tilde{\beta}_{\mathbb{G}})$ are soft subspaces of bornological soft sets $(\mathbb{X},\tilde{\beta}_1)$ and $(\mathbb{Y},\tilde{\beta}_2)$, respectively and ψ is soft bounded mapping from $(\mathbb{X},\tilde{\beta}_1)$ into $(\mathbb{Y},\tilde{\beta}_2)$ such that $\psi(\mathbb{F}) \tilde{\subseteq} \mathbb{G}$. Then ψ is relatively soft bounded mapping of $(\mathbb{F},\tilde{\beta}_{\mathbb{F}})$ into $(\mathbb{G},\tilde{\beta}_{\mathbb{G}})$.

Proof: Let $\tilde{A}' \in \tilde{\beta}_{\mathbb{F}}$, then there is $\tilde{A} \in \tilde{\beta}_1$

$\tilde{A}' = \tilde{A} \tilde{\cap} \mathbb{F}$, $\psi(\tilde{A}) \in \tilde{\beta}_2$, hence $\psi(\tilde{A}') \tilde{\cap} \mathbb{G} = \psi(\tilde{A} \tilde{\cap} \mathbb{F}) \tilde{\cap} \mathbb{G} = \psi(\tilde{A}) \tilde{\cap} \psi(\mathbb{F}) \tilde{\cap} \mathbb{G} \in \tilde{\beta}_{\mathbb{G}}$.

Since ($\psi(\mathbb{F}) \tilde{\subseteq} \mathbb{G}$) $\Longrightarrow$ $\psi(\tilde{A}) \tilde{\cap} \mathbb{G} \in \tilde{\beta}_{\mathbb{G}}$.

Proposition (3-21): Let $(\mathbb{F},\tilde{\beta}_{\mathbb{F}}), (\mathbb{G},\tilde{\beta}_{\mathbb{G}}), (\mathbb{H},\tilde{\beta}_{\mathbb{H}})$ are soft subspaces of a bornological soft sets $(\mathbb{X},\tilde{\beta}_1), (\mathbb{Y},\tilde{\beta}_2), (\mathbb{Z},\tilde{\beta}_3)$ respectively $\psi: (\mathbb{F},\tilde{\beta}_{\mathbb{F}}) \to (\mathbb{G},\tilde{\beta}_{\mathbb{G}})$, $\phi: (\mathbb{G},\tilde{\beta}_{\mathbb{G}}) \to (\mathbb{H},\tilde{\beta}_{\mathbb{H}})$ are two relatively soft bounded maps. If the image for every soft bounded in $\tilde{\beta}_{\mathbb{F}}$ is soft bounded set in $\tilde{\beta}_{\mathbb{G}}$, then the composition $\phi \circ \psi: (\mathbb{F},\tilde{\beta}_{\mathbb{F}}) \to (\mathbb{H},\tilde{\beta}_{\mathbb{H}})$ is relatively soft bounded mapping.

Proof: Let $\tilde{A} \in \tilde{\beta}_{\mathbb{F}}$ and since $\psi: (\mathbb{F},\tilde{\beta}_{\mathbb{F}}) \to (\mathbb{G},\tilde{\beta}_{\mathbb{G}})$ relatively soft bounded mapping.

It follows that

$\psi(\tilde{A}) \tilde{\cap} \mathbb{G} \in \tilde{\beta}_{\mathbb{G}}$.

Again since $\phi: (\mathbb{G},\tilde{\beta}_{\mathbb{G}}) \to (\mathbb{H},\tilde{\beta}_{\mathbb{H}})$. Be relatively soft bounded mapping. It follows that

$\phi\big(\psi(\tilde{A}) \tilde{\cap} \mathbb{G}\big) \tilde{\cap} \mathbb{H} \in \tilde{\beta}_{\mathbb{H}}$ $\phi.\psi\left((\tilde{A}) \tilde{\cap} \mathbb{G}\right) \tilde{\cap} \mathbb{H} \in \tilde{\beta}_{\mathbb{H}}$.

Definition (3-22): Let $(\mathbb{X},\tilde{\beta}_1), (\mathbb{Y},\tilde{\beta}_2)$ are two bornological soft sets. A map $\psi: (\mathbb{X},\tilde{\beta}_1) \to (\mathbb{Y},\tilde{\beta}_2)$ is called a ***soft bornological isomorphism*** if ψ, ψ^{-1} are soft bounded maps and bijective.

Definition (3-23): If $(\mathbb{X},\tilde{\beta}_1)$ and $(\mathbb{Y},\tilde{\beta}_2)$ are bornological soft sets, then the collection of soft bounded sets $\{\tilde{B}_1 \times \tilde{B}_2 : \tilde{B}_1 \in \tilde{\beta}_1, \tilde{B}_2 \in \tilde{\beta}_2\}$ is a soft bornology on $\mathbb{X} \times \mathbb{Y}$ and it is called soft product bornology on $\mathbb{X} \times \mathbb{Y}$ and denoted by $\tilde{\beta}_1 \times \tilde{\beta}_2$. The bornological soft set $(\mathbb{X} \times \mathbb{Y}, \tilde{\beta}_1 \times \tilde{\beta}_2)$ is called ***bornological product soft set***.

Definition (3-24): Let $(\mathbb{X},\tilde{\beta})$ be bornological soft set. A subset $\tilde{A}$ of $\mathbb{X}$ is called ***soft unbounded*** of $\mathbb{X}$ if $\tilde{A}^c \in \tilde{\beta}$.

Proposition (3-25): Suppose $(\mathbb{X},\tilde{\beta})$ is bornological soft set on $\mathbb{X}$ and $\tilde{\Gamma}$ denoted a collection of all soft unbounded sets of $\mathbb{X}$. Then $\tilde{\Gamma}$ satisfies the three following :

i. $\tilde{B}^c = \mathbb{X} - \tilde{B} \in \tilde{\Gamma}$ for each non-empty set $\tilde{B} \tilde{\subseteq} \mathbb{X}$;
ii. If $\tilde{K} \in \tilde{\Gamma}$ and $\tilde{K} \tilde{\subseteq} \tilde{P}$ then $\tilde{P} \in \tilde{\Gamma}$;
iii. The finite intersection of members of $\tilde{\Gamma}$ is also member of $\tilde{\Gamma}$.

Iff, $\mathbb{X} \neq \emptyset$ and $\tilde{\Gamma}$ is the family of subsets of $\mathbb{X}$ such that $\tilde{\Gamma}$ satisfies (i, ii, iii). Then there is a soft bornology $\tilde{\beta}$ on $\mathbb{X}$ such that $\tilde{\Gamma}$ forms the set of all soft unbounded subset of $\mathbb{X}$.

Proof:

i.Let $\tilde{B}$ be a nonempty subset of $\mathbb{X}$, i.e. $\tilde{B} = \cup_{x\in\tilde{B}}\{x\} \in \tilde{\beta}$. Then $\tilde{B}^c = \mathbb{X} - \tilde{B} \in \tilde{\Gamma}$;

ii.Let $\tilde{K} \in \tilde{\Gamma}$ and $\tilde{K} \tilde{\subseteq} \tilde{P} \Longrightarrow \tilde{P}^c \tilde{\subseteq} \tilde{K}^c$.
Since $\tilde{K}^c \in \tilde{\beta}$ (by definition of $\tilde{\Gamma}$) then $\tilde{P}^c \in \tilde{\beta}$ (by condition (ii) of definition soft bornological set) then $\tilde{P} \in \tilde{\Gamma}$;

iii.If $\tilde{K}_1, \tilde{K}_2 \in \tilde{\Gamma}$ then $\tilde{K}_1^c, \tilde{K}_2^c \in \tilde{\beta}$, since $\tilde{\beta}$ is a soft bornology on $\mathbb{X}$ then
$\tilde{K}_1^c \tilde{\cup} \tilde{K}_2^c = (\tilde{K}_1 \tilde{\cap} \tilde{K}_2)^c \in \tilde{\beta}$ then $\tilde{K}_1 \tilde{\cap} \tilde{K}_2 \in \tilde{\Gamma}$.

Conversely, we define $\tilde{\beta} = \{\tilde{B} \tilde{\subseteq} \mathbb{X}: \tilde{B}^c \in \tilde{\Gamma}\}$.

a. By (i) $\mathbb{X} - \{x\} \in \tilde{\Gamma}$ and $\forall\, x \in \mathbb{X}, \{x\} \in \tilde{\beta}$. Then $\tilde{\beta}$ covers $\mathbb{X}$;
b. If $\tilde{B} \in \tilde{\beta}, \tilde{A} \tilde{\subseteq} \tilde{B}$, to prove $\tilde{A} \in \tilde{\beta}$. Since $\tilde{B}^c \tilde{\subseteq} \tilde{A}^c$ (by definition of soft bornological set) $\tilde{B}^c \in \tilde{\Gamma}$ (by (ii)) $\tilde{A}^c \in \tilde{\Gamma}$ then (by definition of soft bornological set) $\tilde{A} \in \tilde{\beta}$;
c. If $\tilde{B}_1, \tilde{B}_2 \in \tilde{\beta}$ (by (iii)) $\tilde{B}_1^c \tilde{\cap} \tilde{B}_2^c \in \tilde{\beta} = \left(\tilde{B}_1 \tilde{\cup} \tilde{B}_2\right)^c \in \tilde{\beta}$.

Theorem (3-26): $(\mathbb{X},\tilde{\beta})$ a bornological soft set, $\mathbb{G} \tilde{\subseteq} \mathbb{X}$ and $(\mathbb{G},\tilde{\beta}_{\mathbb{G}})$ a soft bornological subspace of $\mathbb{X}$. For the sets $\mathbb{H} \tilde{\subseteq} \mathbb{G} \tilde{\subseteq} \mathbb{X}$, the following statements are equivalent:

i. $\mathbb{H} \in \tilde{\Gamma}_{\mathbb{G}}$
ii. $\mathbb{H} = k \tilde{\cap} \mathbb{G}\ for\ \exists\, k \in \tilde{\Gamma}$.

Proof: (i) $\Rightarrow$ (ii) Let $\mathbb{H} \in \tilde{\Gamma}_{\mathbb{G}}$. Then, $(\mathbb{H})^{\tilde{c}}{}_{\mathbb{G}} \in \tilde{\beta}_{\mathbb{G}}$. So, there exists $w \in \tilde{\beta}$ such that $(\mathbb{H})^{\tilde{c}}{}_{\mathbb{G}} = w \tilde{\cap} \mathbb{G}$. We can write

$\mathbb{H} = ((\mathbb{H})^{\tilde{c}}{}_{\mathbb{G}})^{\tilde{c}}{}_{\mathbb{G}} = (w \tilde{\cap} \mathbb{G})^{\tilde{c}}{}_{\mathbb{G}} (w \tilde{\cap} \mathbb{G})^{\tilde{c}}{}_{\mathbb{G}} = \mathbb{G} \tilde{\setminus} (w \tilde{\cap} \mathbb{G})$

$= \mathbb{G} \tilde{\cap} (w \tilde{\cap} \mathbb{G})^{\tilde{c}}{}_{\mathbb{X}} = \mathbb{G} \tilde{\cap} ((w)^{\tilde{c}}{}_{\mathbb{X}} \tilde{\cup} (\mathbb{G})^{\tilde{c}}{}_{\mathbb{X}})$

$= (\mathbb{G} \tilde{\cap} (w)^{\tilde{c}}{}_{\mathbb{X}}) \tilde{\cup} (\mathbb{G} \tilde{\cap} (\mathbb{G})^{\tilde{c}}{}_{\mathbb{X}}) = \mathbb{G} \tilde{\cap} (w)^{\tilde{c}}{}_{\mathbb{X}}$

Since $w \in \tilde{\beta}$ then, $(w)^{\tilde{c}}{}_{\mathbb{X}} \in \tilde{\Gamma}$. By choosing $(w)^{\tilde{c}}{}_{\mathbb{G}} = k$. We obtain that $\mathbb{H} = k \tilde{\cap} \mathbb{G}$ for $k \in \tilde{\Gamma}$.

(ii) $\Rightarrow$ (i) Since $\mathbb{H} = k \tilde{\cap} \mathbb{G}$ then, $(\mathbb{H})^{\tilde{c}}{}_{\mathbb{G}} = (k \tilde{\cap} \mathbb{G})^{\tilde{c}}{}_{\mathbb{G}}$.

It follows that

$(k \tilde{\cap} \mathbb{G})^{\tilde{c}}{}_{\mathbb{G}} = \mathbb{G} \tilde{\setminus} (k \tilde{\cap} \mathbb{G}) = \mathbb{G} \tilde{\cap} (k \tilde{\cap} \mathbb{G})^{\tilde{c}}{}_{\mathbb{X}}$
$= \mathbb{G} \tilde{\cap} ((k)^{\tilde{c}}{}_{\mathbb{X}} \tilde{\cup} (\mathbb{G})^{\tilde{c}}{}_{\mathbb{X}})$
$= (\mathbb{G} \tilde{\cap} (k)^{\tilde{c}}{}_{\mathbb{X}}) \tilde{\cup} (\mathbb{G} \tilde{\cap} (\mathbb{G})^{\tilde{c}}{}_{\mathbb{X}}) = \mathbb{G} \tilde{\cap} (k)^{\tilde{c}}{}_{\mathbb{X}}$
Moreover, by assumption $k \in \tilde{\Gamma}$, then $(k)^{\tilde{c}}{}_{\mathbb{X}} \in \tilde{\beta}$.
Hence, we get $(\mathbb{H})^{\tilde{c}}{}_{\mathbb{G}} \in \tilde{\beta}_{\mathbb{G}}$ and $\mathbb{H} \in \tilde{\Gamma}_{\mathbb{G}}$, as required.

Theorem (3-27): $(\mathbb{X}, \tilde{\beta})$ is bornological soft set, $\mathbb{G} \tilde{\subseteq} \mathbb{X}$ and $(\mathbb{G}, \tilde{\beta}_{\mathbb{G}})$ is bornological soft subspace of $\mathbb{X}$. If $\mathbb{H} \tilde{\subseteq} \mathbb{G}, \mathbb{H} \in \tilde{\Gamma}$ then, $\mathbb{H} \in \tilde{\Gamma}_{\mathbb{G}}$.
Proof: Since $\mathbb{H} \tilde{\subseteq} \mathbb{G}$ then, $\mathbb{H} = \mathbb{H} \tilde{\cap} \mathbb{G}$. By assumption $\mathbb{H} \in \tilde{\Gamma}$ and we get $\mathbb{H} \in \tilde{\Gamma}_{\mathbb{G}}$, , as required.

Theorem (3-28): $(\mathbb{X}, \tilde{\beta})$ is bornological soft set, $\mathbb{G} \tilde{\subseteq} \mathbb{X}$ and $(\mathbb{G}, \tilde{\beta}_{\mathbb{G}})$ is soft bornological subspace of $\mathbb{X}$. The following statements are equivalent:

i. $\mathbb{G} \in \tilde{\Gamma}$
ii. $\tilde{\Gamma}_{\mathbb{G}} \tilde{\subseteq} \tilde{\Gamma}$.

Proof: (i) → (ii) Let $\mathbb{G} \in \tilde{\Gamma}$ and take as given $\mathbb{H} \in \tilde{\Gamma}_{\mathbb{G}}$. So, there exists $k \in \tilde{\Gamma}$ such that $\mathbb{H} = k \tilde{\cap} \mathbb{G}$. Moreover, by assumption $\mathbb{G} \in \tilde{\Gamma}$ then, $\mathbb{H} \in \tilde{\Gamma}$. Thus, $\tilde{\Gamma}_{\mathbb{G}} \tilde{\subseteq} \tilde{\Gamma}$.
(ii) → (i) Since $(\mathbb{G}, \tilde{\beta}_{\mathbb{G}})$ be a soft bornology, then $\mathbb{G} \in \tilde{\Gamma}_{\mathbb{G}}$ and $\tilde{\Gamma}_{\mathbb{G}} \tilde{\subseteq} \tilde{\Gamma}$. Then we get $\mathbb{G} \in \tilde{\Gamma}$.

CONCLUSION

The idea of this research serves as a basis for the future of new studies in the field of bornology. We started by defining of a soft bornology in addition to important examples and theories. We also clarified the soft subspace of the soft bornology and fundamental constructer of it. Also, we generated a space from soft unbounded sets and some results.

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المجموعات الناعمة للولادة

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الخلاصة.

مفهوم المجموعة المحدودة داخل هذه المجموعة أو الفضاء تم إعطائه من قبل العديد من المؤلفين. علم الولادة ، تم تعريفه على مجموعة ناعمة لحل مشاكل الحدود للمجموعة الناعمة. أيضًا ، نقوم ببناء قاعدة ناعمة وقاعدة سفلية ناعمة كجزء من البناء الأساسي للمجموعات الناعمة للولادة. علاوة على ذلك ، من الطبيعي دراسة البناء الأساسي للمجموعات الناعمة للولادة ، مثل الفضاء الجزئي الناعم ، وعلم الولادة الناعم للمنتج والتشابه الناعم للولادات. بالإضافة إلى ذلك ، يمكن أن تكون عائلة مجموعات المواليد الناعمة عبارة عن مجموعة مرتبة جزئية من خلال علاقة مرتبة جزئية ونثبت أن تقاطع المجموعات اللينة المولودة هي مجموعات ناعمة ولادية ولكن اتحاد المجموعات الناعمة للولادة ليس ضروريًا لتكون مجموعات لينة. الترجمة من اليسار إلى اليمين هي تماثل ناعم ، ونتاج المجموعات اللينة المولودة هي مجموعات ناعمة ولادية. أخيرًا ، يتم إنشاء بنية ولادية ناعمة تكون عناصرها مجموعات ناعمة غير محدودة.