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Convergence Analysis of Cubic Spline Function with Fractional Degree and Applications

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ABSTRACT

In this paper, a fractional degree cubic spline scheme is proposed and analyzed for fractional order with the multi-term Riemann–Liouvile (R–L) derivatives. For the integral and fractional differential equations, we handle fractional continuity equations and attain a system of linear algebraic equations by using the matrix method based on piecewise linear test functions. The scheme is to solve the fractional initial value problems to approximate the solution of the fractional equation with spline approximation by using Reimann–Liouvile derivative. In order to obtain a fully discrete method, the standard spline approximation is used to discrete the spatial derivative with continuity conditions that suitable for the scheme method and provided the model is unique and exist for all interval which are appeared in that scheme for the function and all derivatives with fractional order. The convergence analysis is rigorously proved by the spline method. In addition, the existence and uniqueness of numerical solutions for linear systems are proved strictly. Numerical results confirm the theoretical analysis and show the effectiveness of the method.

1. Introduction

The estimation of the theory from the past time experienced. Since the latter half of the 1950s, the rise of spline functions along with advances in computation has accelerated the growth of classical approximation theory, which has developed into a profound theory in mathematics, scientific calculation, engineering technology, etc. See [1-5] a fractional differential equation (FDE) is a generalization of differential equations that replaces integral order derivatives with fractional order derivatives. In general, ordinary differential equations are used to describe dynamic phenomena in several fields such as physics, biology, and chemistry. Though, for some complex systems, ordinary simple differential equations do not give satisfactory results.

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Therefore, FDE is used instead of integer-order models to get better models. See [1,2,3]. The concept of fractions is now thought about a practical technique in many areas of science, including physics (Oldham and Spainer [4, 5]). Srivastava el al. [6] provided a model of under-actuated systems of mechanical by fractional order derivatives, and Sharma [7] reviewed highly generalized fractional equations of kinetic in astrophysics. Caputo [8] again created the more "classical" definition of the fractional Riemann-Liouville derivative and identified integer-order initial conditions for solving his fractional-order differential equations.

Kowankar and Gangel [9] reformulated the broken Riemann-Liouville derivative to differentiate fractional functions that cannot be differentiated anywhere. In [10, 14], the matrix of operational for the left Caputo fraction derivation of the Legendre polynomial basis was presented. Interested readers can also check that in [11] the author of [12] presented the fractional integration operation matrix for his Haar wavelet basis. In [13] constructs the Type equation here.operation matrix for the fractional left Riemann-Liouville integral over the Legendre orthonormal polynomial basis. In the current work, we relate the operational matrix for the fractional Riemann-Liouville integration proposed in [13]. In this paper, a suitable numerical solution was derived based on a fractional polynomial function as a fractional spline basis using the fractional limit of the spline function. Section 2 simulates that the preliminary definitions used to control polynomial splines apply to the solution of fractional differential equations, and Section 3 introduces the formulation of the cubic polynomial spline approximation to the fractional order increase. In Section 4, the theoretical analysis and convergence of the Riemann-Liouville method are performed. Finally, Section 5 presents numerical evidence demonstrating the accuracy of the Maple programming method.

1. Basic Definitions

This section will go over the many definitions of the fractional derivative as well as Taylor's Theorem, which we used in our work. To define, different methods are used. The most common fractional derivatives are the Riemann-Liouville and Caputo derivatives.

Definition 2.1. [22] (Fractional Derivative of Order) The Caputo derivative operator of order α is defined as:

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} \left(\frac{d}{du}\right)^n f(u)du, n = \lceil\alpha\rceil and \alpha > 0.$$

For $a = 0$, we introduce the notation:

$${}^C D_t^\alpha f(t) = D^\alpha f(t).$$

Definition 2.2. [22] (Fractional Derivative of Order) The Riemann-Liouville derivative of order α can be defined as:

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-u)^{n-\alpha-1} f(u)du.$$

For everyα, and $= \lceil\alpha\rceil$.

Definition 2.3.[17] Suppose that $D_a^{z\lambda} G(x) \in \mathbb{C}[a,b]$ for $z = 0,1,\dots,n+1$ and $0 < \lambda \le 1$ then we have the Taylor series expansion about $x = \tau$

$$\boldsymbol{g(x) = \sum_{k=0}^{n} \frac{(x-\tau)^{k\lambda}}{\Gamma(k\lambda+1)} D_a^{k\lambda} g(\tau) + \frac{\left(D_a^{(n+1)\lambda} g\right)(\xi)}{\Gamma((n+1)\lambda+1)} (x-\tau)^{(n+1)\lambda} \text{ with } a \le \xi \le x,}$$

for all $x \in [a,b]$ where $D_a^{z\lambda} = D_a^\lambda \cdot D_a^\lambda \dots D_a^\lambda$ (z times).

Definition 2.3. **[23]** A spline function is a function consisting of polynomial pieces joined together with certain smooth conditions.

The researcher is forced to write:

A function S is a spline of degree k if

1- The domain of S is an interval [a, b].
2- All derivatives *S', S'',…, $S^{(K-1)}$* are all continuous functions on [a, b].
3- There are points t_i (the knots of S) such that a = t_0 < t_1 < t_2 < ….<t_n = b, and such that S is a polynomial of degree at most k on each subinterval [t_i, t_{i+1}]. Smoothness is reflected by the order of the continuous derivative at the knots. If the researcher want the approximating spline to have continuous m-th derivative, the researcher should choose a spline of degree at least m + 1.

2. Description of the Numerical Method

We consider the cubic polynomial spline with fractional degree, by using the fractional continuity derivatives of the formulation of the problems. It's a fantastic method for solving integral and fractional differential equations. In theory, these computations are simple, but in fact, computing the polynomials and showing the convergence of the related series can be challenging.

Theorem 3.1. There exists a unique spline $S(x) \in S_{(n,3)}$, the fractional derivative conditions $D^{(\alpha)} s_j, j = 0,1,\dots,N,$ such that:

$$S(x) = a_i + (x-x_i)b_i + +(x-x_i)^2 c_i + (x-x_i)^3 d_i + (x-x_i)^{\frac{5}{2}} e_i + (x-x_i)^{\frac{7}{2}} f_i . \quad (3.1)$$

The conditions are defined by the following:

$$S(x_i)=y_i \quad , \quad S(x_{i+1})=y_{i+1} \quad ,$$
$$S^{(\frac{1}{2})}(x_{i+1})=y^{(\frac{1}{2})}{}_{i+1} \quad , \qquad (3.2)$$
$$S^{(1)}(x_i)=y^{(1)}{}_i \quad , \quad S^{(1)}(x_{i+1})=y^{(1)}{}_{i+1} \quad ,$$
$$S^{(2)}(x_i)=y^{(2)}{}_i \,.$$

Proof. Let $\{x_i\}_{i=0}^{n}$ be a strictly increasing sequence of points such that $a=x_0<x_1<x_2<\cdots<x_n=b$ consider that $s(x)$ is a continuous function over an interval$[a,b]$, and $s(x)$ be a cubic polynomial spline $s(x)$ of fractional degree such that:

$$S(x_i)=a_i+(x_i-x_i)b_i++(x_i-x_i)^2c_i +(x_i-x_i)^3d_i+(x_i-x_i)^{\frac{5}{2}}e_i+(x_i-x_i)^{\frac{7}{2}}f_i. ,$$

we get the following by using all of the conditions in the above equation and simplified them.

$$y_i=a_i$$

$$, \; y_{i+1}=a_i+hb_i+h^2c_i+h^3d_i+h^{\frac{5}{2}}e_i+h^{\frac{7}{2}}f_i \qquad , y_i^{(1)}=b_i$$

$$, \; c_i=\frac{y_i^{(2)}}{2} \,.$$

Using the above equations a_i, b_i and c_i , we get

$$\frac{-3}{h}y_{i+1}+\frac{3}{h}y_i+2y_i^{(1)}+y_{i+1}^{(1)} +\frac{h}{2}y_i^{(2)}=-\frac{1}{2}h^{\frac{3}{2}}e_i+\frac{1}{2}h^{\frac{5}{2}}f_i,$$

$$f_i=\frac{-900p+2560}{325p-1024}h^{\frac{-7}{2}}y_{i+1}+\frac{160\sqrt{p}}{325p-1024}h^{-3}y_{i+1}^{(\frac{1}{2})} +\frac{300p-1024}{325p-1024}h^{\frac{-5}{2}}y_{i+1}^{(1)}+(\frac{900p-2560}{325p-1024}h^{\frac{-1}{2}} -\frac{160}{325p\sqrt{x_{i+1}}-1024\sqrt{x_{i+1}}})h^{-3}y_i +\frac{600p-1856}{325p-1024}h^{\frac{-5}{2}}y_i^{(1)}+\frac{450p-1408}{975p-3072}h^{\frac{-3}{2}}y_i^{(2)},$$

$$e_i=\frac{1050p-3584}{325p-1024}h^{\frac{-5}{2}}y_{i+1}+\frac{-350p+1024}{325p-1024} h^{\frac{-3}{2}}y_{i+1}^{(1)}+(\frac{-1050p+3584}{325p-1024}h^{\frac{-1}{2}} -\frac{160}{325p\sqrt{x_{i+1}}-1024\sqrt{x_{i+1}}})h^{-2}y_i +\frac{-700p+2240}{325p-1024}h^{\frac{-3}{2}}y_i^{(1)} +\frac{-525p+1664}{975p-3072}h^{\frac{-1}{2}}y_i^{(2)}+\frac{160\sqrt{p}}{325p-1024}h^{-2}y_{i+1}^{(\frac{1}{2})}. \qquad (3.3)$$

Substituting the value e_i in equation (3.3), with using conditions of the equation (3.2), to find the value of d_i in equation (3.1) and after some simplifications, we obtain

$$d_i=\frac{175p}{325p-1024}h^{-3}y_{i+1} +\left[\frac{-175p}{325p-1024}h^{\frac{-1}{2}}+\frac{320}{325p\sqrt{x_{i+1}}-1024\sqrt{x_{i+1}}}\right] h^{\frac{-5}{2}}y_i+\frac{-225p+640}{325p-1024}h^{-2}y_i^{(1)}+\frac{-825p+2560}{1950p-6144}h^{-1}y_i^{(2)} +\frac{-320\sqrt{p}}{325p-1024}h^{\frac{-5}{2}}y_{i+1}^{(\frac{1}{2})}+\frac{50p}{325p-1024}h^{-2}y_{i+1}^{(1)}.$$

After satisfied all known coefficients with the spline function, can be show that the scheme of the spline method are exist and unique.

The theorem's proof is now complete.

Hint. Using the above theorem, we can show that the spline method's model exists and is unique, since it's easily to show that algebraically, after change to linear system. However, the following theorems can be used to show that the spline method construction's convergence analysis is correct.

Lemma 3.2. Let $s(x)\in C^3$the fractional continuity equations, $x_{j+1}=x_j+k\lambda h$,$0\le k\le 1$, with $s(x)$ in $[x_{j-1},x_j]$, then

$D^{(\frac{3}{2})}s(x_j{}^{+})=D^{(\frac{3}{2})}s(x_j{}^{-})$, Respectively for $j=0,1,2,\ldots,N$. (3.4)

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Proof. Leads to the equations (3.1) and using the fractional continuity of the equations (3.3), we obtain

$$S(x_i) = a_i + (x_i - x_i)b_i + +(x_i - x_i)^2 c_i$$
$$+(x_i - x_i)^3 d_i + (x_i - x_i)^{\frac{5}{2}} e_i + (x_i - x_i)^{\frac{7}{2}} f_i .$$

$$S(x) = y_i + (x - x_i)y_i^{(1)} + \frac{1}{2}(x_i - x_i)^2 y_i^{(2)}$$
$$+(x_i - x_i)^3 \left[\begin{array}{l} \frac{175p}{325p - 1024} h^{-3} y_{i+1} \\ +\left(\frac{-175p}{325p - 1024} h^{-\frac{1}{2}} + \frac{320}{325p\sqrt{x_{i+1}} - 1024\sqrt{x_{i+1}}}\right) \\ h^{-\frac{5}{2}} y_i + \frac{-225p + 640}{325p - 1024} h^{-2} y_i^{(1)} \\ + \frac{-825p + 2560}{1950p - 6144} h^{-1} y_i^{(2)} \\ + \frac{-320\sqrt{p}}{325p - 1024} h^{-\frac{5}{2}} y_{i+1}^{(\frac{1}{2})} + \frac{50p}{325p - 1024} h^{-2} y_{i+1}^{(1)}. \end{array} \right]$$
$$+(x_i - x_i)^{\frac{5}{2}} \left[\begin{array}{l} \frac{1050p - 3584}{325p - 1024} h^{-\frac{5}{2}} y_{i+1} + \frac{-350p + 1024}{325p - 1024} \\ h^{-\frac{3}{2}} y_{i+1}^{(1)} \\ + \left(\frac{-1050p + 3584}{325p - 1024} h^{-\frac{1}{2}} - \frac{160}{325p\sqrt{x_{i+1}} - 1024\sqrt{x_{i+1}}} \right) \\ h^{-2} y_i + \frac{-700p + 2240}{325p - 1024} h^{-\frac{3}{2}} y_i^{(1)} \\ + \frac{-525p + 1664}{975p - 3072} h^{-\frac{1}{2}} y_i^{(2)} + \frac{160\sqrt{p}}{325p - 1024} h^{-2} y_{i+1}^{(\frac{1}{2})}. \end{array} \right]$$
$$+(x_i - x_i)^{\frac{7}{2}} \left[\begin{array}{l} \frac{-900p + 2560}{325p - 1024} h^{-\frac{7}{2}} y_{i+1} + \frac{160\sqrt{p}}{325p - 1024} h^{-3} y_{i+1}^{(\frac{1}{2})} \\ + \frac{300p - 1024}{325p - 1024} h^{-\frac{5}{2}} y_{i+1}^{(1)} + \left(\frac{900p - 2560}{325p - 1024} h^{-\frac{1}{2}} \right. \\ \left. - \frac{160}{325p\sqrt{x_{i+1}} - 1024\sqrt{x_{i+1}}} \right) h^{-3} y_i \\ + \frac{600p - 1856}{325p - 1024} h^{-\frac{5}{2}} y_i^{(1)} + \frac{450p - 1408}{975p - 3072} h^{-\frac{3}{2}} y_i^{(2)} \end{array} \right]. \quad (3.5)$$

$$S^{(\frac{3}{2})}(x_{i+1}) = \frac{b_i}{\sqrt{p x_{i+1}}} + \frac{4}{\sqrt{p}} c_i (x_{i+1} - x_i)^{\frac{1}{2}}$$
$$+ \frac{8}{\sqrt{p}} d_i (x_{i+1} - x_i)^{\frac{3}{2}} + \frac{15}{8}\sqrt{p} e_i (x_{i+1} - x_i)$$
$$+ \frac{105}{8}\sqrt{p} f_i (x_{i+1} - x_i)^2 .$$

$$S^{(\frac{3}{2})}(x_{i+1}) = \frac{b_i}{\sqrt{p x_{i+1}}} + \frac{4}{\sqrt{p}} c_i h^{\frac{1}{2}} + \frac{8}{\sqrt{p}} d_i h^{\frac{3}{2}}$$
$$+ \frac{15}{8}\sqrt{p} e_i h + \frac{105}{8}\sqrt{p} f_i h^2 .$$

$$\frac{b_i}{\sqrt{p x_{i+1}}} + \frac{4}{\sqrt{p}} c_i h^{\frac{1}{2}} + \frac{8}{\sqrt{p}} d_i h^{\frac{3}{2}} + \frac{15}{8}\sqrt{p} e_i h$$
$$+ \frac{105}{8}\sqrt{p} f_i h^2 = \frac{b_{i-1}}{\sqrt{p x_{i+1}}} + \frac{4}{\sqrt{p}} c_{i-1} h^{\frac{1}{2}}$$
$$+ \frac{8}{\sqrt{p}} d_{i-1} h^{\frac{3}{2}} + \frac{15}{8}\sqrt{p} e_{i-1} h + \frac{105}{8}\sqrt{p} f_{i-1} h^2 ,$$

$$\frac{1}{\sqrt{p x_{i+1}}} a_1 h^{-1}\left[y_i - y_{i-1} \right] + a_2 h^{-\frac{3}{2}} \left[y_{i-1} - 2y_i + y_{i+1} \right]$$
$$= \frac{-1}{\sqrt{p x_{i+1}}} \left[y_i^{(1)} - y_{i-1}^{(1)} \right] + b_1 h^{-\frac{1}{2}} \left[y_i^{(1)} - y_{i-1}^{(1)} \right] + b_2 h^{-\frac{1}{2}}$$
$$\left[y_{i+1}^{(1)} - y_i^{(1)} \right] + g h^{\frac{1}{2}} \left[y_i^{(2)} - y_{i-1}^{(2)} \right] + a_1 h^{-1} \left[y_{i+1}^{(\frac{1}{2})} - y_i^{(\frac{1}{2})} \right],$$

where

$$\alpha_1 = \frac{2560 - 825\pi}{325\pi - 1024}, \quad \alpha_2 = \frac{3080\pi - 984.375\pi^2}{325\pi\sqrt{\pi} - 1024\sqrt{\pi}},$$
$$b_1 = \frac{-5120 + 3690p - 656.25p^2}{325p\sqrt{p} - 1024\sqrt{p}},$$
$$\beta_2 = \frac{1040\pi - 328.125\pi^2}{325\pi\sqrt{\pi} - 1024\sqrt{\pi}}, \quad \gamma_1 = \frac{-8192 + 5700\pi - 984.375\pi^2}{1950\pi\sqrt{\pi} - 6144\sqrt{\pi}}.$$

This linear system can be written as follows:

$$\left(\frac{-1}{\sqrt{p x_{i+1}}} a_1 h^{-1} + a_2 h^{-\frac{3}{2}} \right) y_{i-1} + \left(\frac{1}{\sqrt{p x_{i+1}}} a_1 h^{-1} - 2a_2 h^{-\frac{3}{2}} \right) y_i$$
$$+ a_2 h^{-\frac{3}{2}} y_{i+1} = \left(\frac{1}{\sqrt{p x_{i+1}}} - b_1 h^{-\frac{1}{2}} \right) y_{i-1}^{(1)}$$
$$+ \left(\frac{-1}{\sqrt{p x_{i+1}}} + (b_1 - b_2) h^{-\frac{1}{2}} \right) y_i^{(1)} + b_2 h^{-\frac{1}{2}} y_{i+1}^{(1)} - g h^{\frac{1}{2}} y_{i-1}^{(2)}$$
$$+ g h^{\frac{1}{2}} y_i^{(2)} - a_1 h^{-1} y_i^{(\frac{1}{2})} + a_1 h^{-1} y_{i+1}^{(\frac{1}{2})} .$$

$$Y(x) = \left[y_{i-1}, y_i, y_{i+1}, \ldots \right]^T ,$$
$$\bar{Y}(x) = \left[y_{i-1}^{(1)}, y_i^{(1)}, y_{i+1}^{(1)}, \ldots \right]^T .$$

$$\hat{Y}(x) = \left[y_{i-1}^{(2)}, y_i^{(2)}, \ldots\right]^T ,$$

$$\widetilde{Y}(x) = \left[y_i^{(\frac{1}{2})}, y_{i+1}^{(\frac{1}{2})}, \ldots \right]^T .$$

$$AY(x) = B\overline{Y}(x) + C\hat{Y}(x) + D\widetilde{Y}(x) .$$

$$A = \begin{pmatrix} \frac{-1}{\sqrt{\pi x_{i+1}}}\alpha_1 h^{-1} + \alpha_2 h^{\frac{-3}{2}} & \frac{1}{\sqrt{\pi x_{i+1}}}\alpha_1 h^{-1} - 2\alpha_2 h^{\frac{-3}{2}} & \alpha_2 h^{\frac{-3}{2}} & \cdots & 0 \\ 0 & \frac{-1}{\sqrt{\pi x_{i+1}}}\alpha_1 h^{-1} + \alpha_2 h^{\frac{-3}{2}} & \frac{1}{\sqrt{\pi x_{i+1}}}\alpha_1 h^{-1} - 2\alpha_2 h^{\frac{-3}{2}} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{-1}{\sqrt{\pi x_{i+1}}}\alpha_1 h^{-1} + \alpha_2 h^{\frac{-3}{2}} & \frac{1}{\sqrt{\pi x_{i+1}}}\alpha_1 h^{-1} - 2\alpha_2 h^{\frac{-3}{2}} \\ 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{\pi x_{i+1}}}\alpha_1 h^{-1} + \alpha_2 h^{\frac{-3}{2}} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{\pi x_{i+1}}} - \beta_1 h^{\frac{-1}{2}} & \frac{-1}{\sqrt{\pi x_{i+1}}} + (\beta_1 - \beta_2) h^{\frac{-1}{2}} & \beta_2 h^{\frac{-3}{2}} & \cdots & 0 \\ 0 & \frac{1}{\sqrt{\pi x_{i+1}}} - \beta_1 h^{\frac{-1}{2}} & \frac{-1}{\sqrt{\pi x_{i+1}}} + (\beta_1 - \beta_2) h^{\frac{-1}{2}} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{\pi x_{i+1}}} - \beta_1 h^{\frac{-1}{2}} & \frac{-1}{\sqrt{\pi x_{i+1}}} + (\beta_1 - \beta_2) h^{\frac{-1}{2}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{\pi x_{i+1}}} - \beta_1 h^{\frac{-1}{2}} \end{pmatrix},$$

$$C = \begin{pmatrix} -\gamma_1 h^{\frac{1}{2}} & \gamma_1 h^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & -\gamma_1 h^{\frac{1}{2}} & \gamma_1 h^{\frac{1}{2}} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\gamma_1 h^{\frac{1}{2}} & \gamma_1 h^{\frac{1}{2}} \\ 0 & 0 & \cdots & 0 & -\gamma_1 h^{\frac{1}{2}} \end{pmatrix},$$

$$D = \begin{pmatrix} -\alpha_1 h^{-1} & \alpha_1 h^{-1} & 0 & \cdots & 0 \\ 0 & -\alpha_1 h^{-1} & \alpha_1 h^{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\alpha_1 h^{-1} & \alpha_1 h^{-1} \\ 0 & 0 & \cdots & 0 & -\alpha_1 h^{-1} \end{pmatrix}.$$

3. Convergence Analysis

In this section, we'll look at the convergence analysis of the proposed cubic spline method with fractional order, equations (3.1) and (3.5) are considered for evaluating the equation's using fractional differential equations, and is dedicated to the convergence of the spline method.

Theorem 4.1. Let $y(x) \in C^3[0,h]$ be the exact function, and $s(x)$ be a fractional polynomial that exists for all points x in [a, b], and that matches $y(x)$ and its first n-1 derivatives $y^{(r)}$at 0 and h. Then

$$\left\|S^{(\phi)}(x_{i+1}) - y^{(\phi)}(x)\right\| \le \begin{cases} \frac{16}{105\sqrt{\pi}} h^{\frac{7}{2}} \omega_\vartheta(h) & , \quad \phi = 0 \\ \frac{1}{6} h^3 \omega_\vartheta(h) & , \quad \phi = \frac{1}{2} \\ \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} \omega_\vartheta(h) & , \quad \phi = 1 \\ \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} \omega_\vartheta(h) & , \quad \phi = 2 \end{cases}$$

where $\omega_\vartheta(h) = \left\|S^{(\frac{7}{2})}(\beta_a) - y^{(\frac{7}{2})}(\alpha_a)\right\|$ the modulus of continuity in [24] , $\beta_a, \alpha_a \in [0,1]$, and ϑ is the order of truncation errors.

Proof. First the truncation errors of order $\vartheta = 0$, since $S(x)$ is the non-polynomial spline of fractional order and a piecewise continuous function on $[a,b]$, for $j = 1,2,\ldots,n$. Suppose that $y^{(\phi)}(x)$ denoted the restriction of $S^{(\phi)}(x)$ over $\left[x_{j-1}, x_j\right]$ then for J is zero, we have

$$S(x_{i+1}) = y_i + \frac{2}{\sqrt{p}} h^{\frac{1}{2}} y_i^{(\frac{1}{2})} + h y_i^{(1)} + \frac{4}{3\sqrt{p}} h^{\frac{3}{2}} y_i^{(\frac{3}{2})}$$
$$+ \frac{1}{2!} h^2 y_i^{(2)} + \frac{8}{15\sqrt{p}} h^{\frac{5}{2}} y_i^{(\frac{5}{2})}$$
$$+ \frac{1}{3!} h^3 y_i^{(3)} + \frac{16}{105\sqrt{p}} h^{\frac{7}{2}} S_i^{(\frac{7}{2})}(b_a).$$

$$\left|S(x_{i+1}) - y_{i+1}\right| \le \Big(y_i + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(\frac{1}{2})} + hy_i^{(1)} + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{(\frac{3}{2})} + \frac{1}{2!}h^2y_i^{(2)} + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{(\frac{5}{2})} + \frac{1}{3!}h^3y_i^{(3)} + \frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}S_i^{(\frac{7}{2})}(\beta_a)\Big)$$
$$- \Big(y_i + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(\frac{1}{2})} + hy_i^{(1)} + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{(\frac{3}{2})} + \frac{1}{2!}h^2y_i^{(2)} + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{(\frac{5}{2})} + \frac{1}{3!}h^3y_i^{(3)} + \frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}y_i^{(\frac{7}{2})}(\alpha_a)\Big)$$

$$\left|S(x_{i+1}) - y_{i+1}\right| \le \frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}\left\|S^{(\frac{7}{2})}(\beta_a) - y^{(\frac{7}{2})}(\alpha_a)\right\|.$$

for $\emptyset = 1/2$, we obtain

$$\left|S^{(\frac{1}{2})}(x_{i+1}) - y_{i+1}^{(\frac{1}{2})}\right| \le \Big(y_i^{(\frac{1}{2})} + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(1)} + hy_i^{(\frac{3}{2})} + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{(2)} + \frac{1}{2!}h^2y_i^{(\frac{5}{2})} + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{(3)} + \frac{1}{3!}h^3S_i^{(\frac{7}{2})}(\beta_a)\Big) - \Big(y_i^{(\frac{1}{2})} + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(1)}$$
$$+ hy_i^{(\frac{3}{2})} + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{(2)} + \frac{1}{2!}h^2y_i^{(\frac{5}{2})} + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{(3)} + \frac{1}{3!}h^3y_i^{(\frac{7}{2})}(\alpha_a)\Big).$$

$$\left|S^{(\frac{1}{2})}(x_{i+1}) - y_{i+1}^{(\frac{1}{2})}\right| \le \frac{1}{3!}h^3\left\|S^{(\frac{7}{2})}(\beta_a) - y^{(\frac{7}{2})}(\alpha_a)\right\| ,$$

for $\emptyset = 1$, we obtain

$$\left|S^{(1)}(x_{i+1}) - y_{i+1}^{(1)}\right| \le \Big(y_i^{(1)} + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(\frac{3}{2})} + hy_i^{(2)} + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{(\frac{5}{2})} + \frac{1}{2!}h^2y_i^{(3)} + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}S_i^{(\frac{7}{2})}(\beta_a)\Big)$$
$$- \Big(y_i^{(1)} + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(\frac{3}{2})} + hy_i^{(2)} + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{(\frac{5}{2})} + \frac{1}{2!}h^2y_i^{(3)} + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}y_i^{(\frac{7}{2})}(\alpha_a)\Big)$$

$$\left|S^{(1)}(x_{i+1}) - y_{i+1}^{(1)}\right| \le \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}\left\|S^{(\frac{7}{2})}(\beta_a) - y^{(\frac{7}{2})}(\alpha_a)\right\| ,$$

for $\emptyset = 2$, we obtain

$$\left|S^{(2)}(x_{i+1}) - y_{i+1}^{(2)}\right| \le \left(y_i^{(2)} + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(\frac{5}{2})} + hy_i^{(3)} + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}S_i^{(\frac{7}{2})}(\beta_a)\right)$$
$$-\left(y_i^{(2)} + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(\frac{5}{2})} + hy_i^{(3)} + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{(\frac{7}{2})}(\alpha_a)\right)$$

$$\left|S^{(2)}(x_{i+1}) - y_{i+1}^{(2)}\right| \le \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left\|S^{(\frac{7}{2})}(\beta_a) - y^{(\frac{7}{2})}(\alpha_a)\right\|.$$

Theorem 4.2. Let $y(x) \in C^{\alpha}[0,h]$ be the exact function, $s(x)$ be a fractional derivatives of scheme spline function that exists for all points x in [a, b], and that matches $y(x)$ and its first α -1 fractional derivatives $y^{(\alpha)}$at 0 and h. Then

$$\left\|S^{(\phi)}(x_{i+1}) - y^{(\phi)}(x)\right\| \le \begin{cases} a_1\|y(\lambda)\| + b_1\|y^{(1)}(\lambda)\| + c_1\left\|y^{(\frac{1}{2})}(\lambda)\right\| + +d_1\left\|y^{(\frac{3}{2})}(\lambda)\right\| + \frac{1}{2}h^2\omega_{\vartheta}(h) & , \quad \phi = \frac{3}{2} \\ a_2\|y(\lambda)\| + b_1\|y^{(2)}(\lambda)\| + c_2\left\|y^{(\frac{1}{2})}(\lambda)\right\| + +d_2\left\|y^{(\frac{3}{2})}(\lambda)\right\| + h\omega_{\vartheta}(h) & , \quad \phi = \frac{5}{2} \\ a_3\|y(\lambda)\| + c_3\left\|y^{(\frac{1}{2})}(\lambda)\right\| + +d_3\left\|y^{(\frac{3}{2})}(\lambda)\right\| + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}\omega_{\vartheta}(h) & , \quad \phi = 3 \\ a_4\|y(\lambda)\| + c_4\left\|y^{(\frac{1}{2})}(\lambda)\right\| + \omega_{\vartheta}(h) & , \quad \phi = \frac{7}{2} \end{cases}$$

Where

$$a_1 = \frac{2560 - 825\pi}{325\pi\sqrt{\pi x_{i+1}} - 1024\sqrt{\pi x_{i+1}}}h^{-1}, \quad b_1 = \frac{1}{\sqrt{\pi x_{i+1}}},$$
$$c_1 = \frac{3600 - 1143.75\pi}{325\pi - 1024}h^{-1} \quad , \; d_1 = \frac{1472 - 468.75\pi}{975\pi - 3072}$$

$$a_2=\frac{3840-1350\pi}{325\pi\sqrt{\pi x_{i+1}}-1024\sqrt{\pi x_{i+1}}}h^{-2},$$

$$c_2=\frac{20520-6525\pi}{325\pi-1024}h^{-2},\quad d_2=\frac{4000-1275\pi}{325\pi-1024}h^{-1},$$

$$a_3=\frac{-180}{325\pi\sqrt{\pi x_{i+1}}-1024\sqrt{\pi x_{i+1}}}h^{\frac{-5}{2}},$$

$$c_3=\frac{67200-21345\pi}{325\pi\sqrt{\pi}-1024\sqrt{\pi}}h^{\frac{-5}{2}},$$

$$d_3=\frac{17920-5695\pi}{325\pi\sqrt{\pi}-1024\sqrt{\pi}}h^{\frac{-3}{2}}$$

$$a_4=\frac{-1050\sqrt{\pi}}{325\pi\sqrt{x_{i+1}}-1024\sqrt{x_{i+1}}}h^{-3},$$

$$c_4=\frac{33600-10762.5\pi}{325\pi-1024}h^{-3},\ \text{where } \alpha_a,\beta_a\in(a,b).$$

Proof: Since $D^{\alpha}s(x)$ is spline interpolation polynomial of degree 3, and matching

$$y^{(\frac{3}{2})}(x_{i+1})=y_i^{(\frac{3}{2})}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(2)}+hy_i^{(\frac{5}{2})}+\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}y_i^{(3)}+\frac{1}{2!}h^2y_i^{(\frac{7}{2})}(\alpha_a)$$

$$S^{(\frac{3}{2})}(x_{i+1})=\frac{b_i}{\sqrt{\pi x_{i+1}}}+\frac{4}{\sqrt{\pi}}h^{\frac{1}{2}}c_i+\frac{8}{\sqrt{\pi}}h^{\frac{3}{2}}d_i+\frac{15}{8}\sqrt{\pi}he_i+\frac{105}{8}\sqrt{\pi}h^2f_i$$

Write $y_{i+1},y_{i+1}^{(\frac{1}{2})}$ and $y_{i+1}^{(1)}$ in Taylor series form , we get:

$$S^{(\frac{3}{2})}(x_{i+1})=\frac{1}{\sqrt{px_{i+1}}}y_i^{(1)}+\frac{2}{\sqrt{p}}h^{\frac{1}{2}}y_i^{(2)}$$
$$+\frac{2560-825p}{325p\sqrt{px_{i+1}}-1024\sqrt{px_{i+1}}}h^{-1}y_i$$
$$+\frac{4}{3\sqrt{p}}h^{\frac{3}{2}}y_i^{(3)}+hy_i^{(\frac{5}{2})}+\frac{3600-1143.75p}{325p-1024}h^{-1}y_i^{(\frac{1}{2})}$$
$$+\frac{-1600+506.25p}{975p-3072}y_i^{(\frac{3}{2})}+\frac{1}{2}h^2S^{(\frac{7}{2})}(b_a).$$

$$\left|S^{(\frac{3}{2})}(x_{i+1})-y_{i+1}^{(\frac{3}{2})}\right|\le(\frac{1}{\sqrt{px_{i+1}}}y_i^{(1)}+\frac{2}{\sqrt{p}}h^{\frac{1}{2}}y_i^{(2)}$$
$$+\frac{2560-825p}{325p\sqrt{px_{i+1}}-1024\sqrt{px_{i+1}}}h^{-1}y_i+\frac{4}{3\sqrt{p}}h^{\frac{3}{2}}y_i^{(3)}$$
$$+hy_i^{(\frac{5}{2})}+\frac{3600-1143.75p}{325p-1024}h^{-1}y_i^{(\frac{1}{2})}$$
$$+\frac{-1600+506.25p}{975p-3072}y_i^{(\frac{3}{2})}+\frac{1}{2}h^2S^{(\frac{7}{2})}(b_a))-(y_i^{(\frac{3}{2})}$$
$$+\frac{2}{\sqrt{p}}h^{\frac{1}{2}}y_i^{(2)}+hy_i^{(\frac{5}{2})}+\frac{4}{3\sqrt{p}}h^{\frac{3}{2}}y_i^{(3)}+\frac{1}{2!}h^2y_i^{(\frac{7}{2})}(a_a)).$$

$$y^{(\frac{5}{2})}(x_{i+1})=y_i^{(\frac{5}{2})}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(3)}+hy_i^{(\frac{7}{2})}(\alpha_a).$$

$$S^{(\frac{5}{2})}(x_{i+1})=\frac{2}{\sqrt{\pi x_{i+1}}}c_i+\frac{12}{\sqrt{\pi}}h^{\frac{1}{2}}d_i+\frac{15}{8}\sqrt{\pi}e_i+\frac{105}{16}\sqrt{\pi}hf_i$$

Using the values of c_i,d_i,e_i and f_i, to obtain

$$S^{(\frac{5}{2})}(x_{i+1})=(\frac{1}{\sqrt{px_{i+1}}}h^{\frac{1}{2}}+\frac{30720-22140p+3937.5p^2}{1950p\sqrt{p}-6144\sqrt{p}})$$
$$h^{\frac{-1}{2}}y_i^{(2)}+\frac{12180\sqrt{p}-3937.5p\sqrt{p}}{325p-1024}h^{\frac{-5}{2}}y_{i+1}$$
$$+(\frac{-12180\sqrt{p}+3937.5p\sqrt{p}}{325p-1024}h^{\frac{-1}{2}}+\frac{3840-1350p}{325p\sqrt{px_{i+1}}-1024\sqrt{px_{i+1}}})$$
$$h^2y_i+\frac{7680-10680p+2625p^2}{325p\sqrt{p}-1024\sqrt{p}}h^{\frac{-3}{2}}y_i^{(1)}$$
$$+\frac{-3840+1350p}{325p-1024}h^{-2}y_{i+1}^{(\frac{1}{2})}+\frac{-4200\sqrt{p}+1312.5p\sqrt{p}}{325p-1024}h^{\frac{-3}{2}}y_{i+1}^{(1)}.$$

$$\left|S^{(\frac{5}{2})}(x_{i+1})-y^{(\frac{5}{2})}(x_{i+1})\right|\le(\frac{1}{\sqrt{px_{i+1}}}y_i^{(2)}+$$
$$\frac{3840-1350p}{325p\sqrt{px_{i+1}}-1024\sqrt{px_{i+1}}}h^{-2}y_i+y_i^{(\frac{5}{2})}$$
$$+\frac{2}{\sqrt{p}}h^{\frac{1}{2}}y_i^{(3)}+\frac{20520-6525p}{325p-1024}h^{-2}y_i^{(\frac{1}{2})}$$
$$+\frac{400-1275p}{325p-1024}h^{-1}y_i^{(\frac{3}{2})}+hS^{(\frac{7}{2})}(b_a))$$
$$-(y_i^{(\frac{5}{2})}+\frac{2}{\sqrt{p}}h^{\frac{1}{2}}y_i^{(3)}+hy_i^{(\frac{7}{2})}(a_a)).$$

$$\left|S^{(\frac{5}{2})}(x_{i+1})-y^{(\frac{5}{2})}(x_{i+1})\right| \le \frac{3840-1350\pi}{325\pi\sqrt{\pi x_{i+1}}-1024\sqrt{\pi x_{i+1}}} h^{-2}y_i+\frac{1}{\sqrt{\pi x_{i+1}}}y_i^{(2)}+\frac{20520-6525\pi}{325\pi-1024}h^{-2}y_i^{(\frac{1}{2})}+\frac{400-1275\pi}{325\pi-1024}h^{-1}y_i^{(\frac{3}{2})}+h\left\|S^{(\frac{7}{2})}(b_a)-y^{(\frac{7}{2})}(a_a)\right\|.$$

$$S^{(3)}(x_{i+1})=\frac{33600-10762.5\pi}{325\pi-1024}h^{-3}y_{i+1}+\Big(\frac{-33600+10762.5\pi}{325\pi-1024}h^{\frac{-1}{2}}+\frac{-180}{325\pi\sqrt{\pi x_{i+1}}-1024\sqrt{\pi x_{i+1}}}\Big)h^{\frac{-5}{2}}y_i+\frac{-20520+6525\pi}{325\pi-1024}h^{-2}y_i^{(1)}+\frac{-21600+6862.5\pi}{325\pi-1024}h^{-1}y_i^{(2)}+\frac{180\sqrt{\pi}}{325\pi-1024}h^{\frac{-5}{2}}y_{i+1}^{(\frac{1}{2})}+\frac{-13440+4237.5\pi}{325\pi-1024}h^{-2}y_{i+1}^{(2)}.$$

$$\left|S^{(3)}(x_{i+1})-y^{(3)}(x_{i+1})\right| \le \Big(\frac{-180}{325\pi\sqrt{\pi x_{i+1}}-1024\sqrt{\pi x_{i+1}}}h^{\frac{-5}{2}}y_i+\frac{67200-21345\pi}{325\pi\sqrt{\pi}-1024\sqrt{\pi}}h^{\frac{-5}{2}}y_i^{(\frac{1}{2})}+\frac{17920-5695\pi}{325\pi\sqrt{\pi}-1024\sqrt{\pi}}h^{\frac{-3}{2}}y_i^{(\frac{3}{2})}+y_i^{(3)}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}S^{(\frac{7}{2})}(b_a)\Big)-\Big(y_i^{(3)}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}y_i^{(\frac{7}{2})}(a_a)\Big).$$

$$\left|S^{(3)}(x_{i+1})-y^{(3)}(x_{i+1})\right| \le \frac{-180}{325\pi\sqrt{\pi x_{i+1}}-1024\sqrt{\pi x_{i+1}}}h^{\frac{-5}{2}}y_i+\frac{67200-21345\pi}{325\pi\sqrt{\pi}-1024\sqrt{\pi}}h^{\frac{-5}{2}}y_i^{(\frac{1}{2})}+\frac{17920-5695\pi}{325\pi\sqrt{\pi}-1024\sqrt{\pi}}h^{\frac{-3}{2}}y_i^{(\frac{3}{2})}+\frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}\left\|S^{(\frac{7}{2})}(b_a))-y^{(\frac{7}{2})}(a_a)\right\|.$$

$$S^{(\frac{7}{2})}(x_{i+1})=\Big(\frac{5906.25\pi\sqrt{\pi}-16800\sqrt{\pi}}{325\pi-1024}h^{\frac{-1}{2}}+\frac{-1050\sqrt{\pi}}{325\pi\sqrt{\pi x_{i+1}}-1024\sqrt{\pi x_{i+1}}}\Big)h^{-3}y_i+\frac{3937.5\pi\sqrt{\pi}-12180\sqrt{\pi}}{325\pi-1024}h^{\frac{-5}{2}}y_i^{(1)}+\frac{2953.123\pi\sqrt{\pi}-9240\sqrt{\pi}}{925\pi-3072}h^{\frac{-3}{2}}y_i^{(2)}+\frac{-5906.25\pi\sqrt{\pi}+16800\sqrt{\pi}}{325\pi-1024}h^{\frac{-7}{2}}y_{i+1}+\frac{1050\sqrt{\pi}}{325\pi-1024}h^{-3}y_{i+1}^{(\frac{1}{2})}+\frac{1968\pi\sqrt{\pi}-6720\sqrt{\pi}}{325\pi-1024}h^{\frac{-5}{2}}y_{i+1}^{(1)}.$$

$$\left|S^{(\frac{7}{2})}(x_{i+1})-y^{(\frac{7}{2})}(x_{i+1})\right| \le \frac{-1050\sqrt{\pi}}{325\pi\sqrt{\pi x_{i+1}}-1024\sqrt{\pi x_{i+1}}}\Big)h^{-3}y_i+\frac{33600-10762.5\pi}{325\pi\sqrt{\pi}-1024\sqrt{\pi}}h^{-3}y_i^{(\frac{1}{2})}+\left\|S^{(\frac{7}{2})}(b_a))-y^{(\frac{7}{2})}(a_a)\right\|.$$

The theorem's proof is now complete.

4. Numerical Results and Discussion

In the two numerical examples in this section, the method is used to complete all computations; two problems are considered to define the class C^3 of fractional interpolation spline and to test the computational applicability of the provided method. The application of the results in two parts shows the value of the proposed technique. Tables 1, 2 and 3 are given below.

The term $e, e^{\left(\frac{1}{2}\right)}$ and $e^{(1)}$ represent the maximum magnitude errors $|e(x)| = |s(x) - y(x)|$, $\left|D^{\left(\frac{1}{2}\right)}e(x)\right| = \left|D^{\left(\frac{1}{2}\right)}s(x) - D^{\left(\frac{1}{2}\right)}y(x)\right|$, and $|e'(x)| = |S'(x) - y'(x)|$ respectively, and the First error $L_1 = \sum_{i=1}^{n}|s_i(x) - y(x)|$, the Euclidian error $L_2 = \sqrt{\sum_{i=1}^{n}|s_i(x) - y(x)|^2}$ and the maximum error $L_\infty = \max_{a<x<b}|s_i(x) - y(x)|$ see [23].

Example 5.1 [24]Consider the following nonlineanr FIVP as

$$D^{\frac{3}{2}}y(t)+y^2(t)=\frac{\Gamma(6)}{\Gamma(4.5)}t^{\frac{7}{2}}-\frac{3\Gamma(5)}{\Gamma(3.5)}t^{\frac{5}{2}}+\frac{\Gamma(4)}{\Gamma(2.5)}t^{\frac{3}{2}}+[t^5-3t^4+2t^3]^2$$

and the $y'(0)=0$, $y(0)=0$ with initial condition . $y(t)=t^5-3t^4+2t^3$ exact solution is

Table 1 Absolute error of $S(x)$, and its derivative of example 5.1.

h	$L_\infty - error$	$S^{(\frac{1}{2})}(x) - y^{(\frac{1}{2})}(x)$	$S'(x) - y'(x)$	$L_1 - error$	$L_2 - error$
0.001	9.61×10^{-12}	1.06×10^{-8}	6.07×10^{-7}	4.22×10^{-11}	2.40×10^{-3}
0.002	8.46×10^{-11}	6.11×10^{-8}	2.47×10^{-6}	3.50×10^{-10}	4.09×10^{-3}
0.003	3.09×10^{-10}	1.71×10^{-7}	5.64×10^{-6}	1.22×10^{-9}	5.60×10^{-3}
0.004	7.86×10^{-10}	3.57×10^{-7}	1.02×10^{-5}	2.99×10^{-9}	7.02×10^{-3}
0.01	1.56×10^{-8}	4.84×10^{-6}	6.93×10^{-5}	5.41×10^{-8}	1.46×10^{-2}
0.02	1.20×10^{-7}	2.44×10^{-5}	3.12×10^{-4}	4.96×10^{-7}	2.53×10^{-2}
0.03	1.88×10^{-7}	7.39×10^{-5}	7.70×10^{-4}	1.74×10^{-6}	3.41×10^{-2}
0.04	5.54×10^{-7}	1.63×10^{-4}	1.48×10^{-3}	4.06×10^{-6}	4.10×10^{-2}

Table 2 Comparison with the method in [24]. The absolute error in Example 5.1 is shown.

h	Our method	Ref[24]
0.001	9.61×10^{-12}	2.2062×10^{-11}
0.05	3.98×10^{-6}	1.1743×10^{-4}
0.02	1.20×10^{-7}	3.3143×10^{-6}

The maximum absolute error in [24] with 2.2062×10^{-11} is analyzed in which has result $h = 0.0001$ while the maximum absolute error using our method which is presented in table 1 and table 2. The numerical results of this problem show that our method converges exponentially and is more accurate than the method [24].

Example 5.2 [19] Consider the following initial value problem $D^2u(x) + \sin(x)D^{\frac{1}{2}}u(x) + x\,u(x) = f(x)$, $x \in [0,1]$ where $f(x) = x^9 - x^8 + 56x^6 - 42x^5 + \sin(x)(\frac{32768}{6435}x^{\frac{15}{2}} - \frac{2048}{429}x^{\frac{13}{2}})$ with initial conditions $u(0) = 0$ and $u'(0) = 0$ The exact solution is $u(x) = x^8 - x^7$.

Table 3 Absolute error of $S(x)$ and its derivative of example 5.2.

h	$L_\infty - error$	$S^{(\frac{1}{2})}(x) - y^{(\frac{1}{2})}(x)$	$S'(x) - y'(x)$	$L_1 - error$	$L_2 - error$
0.001	2.5482×10^{-26}	2.1×10^{-29}	1.206×10^{-27}	5.2396×10^{-27}	2.2481×10^{-7}
0.002	3.6451×10^{-23}	2.1588×10^{-26}	8.7149×10^{-25}	7.5034×10^{-24}	1.3833×10^{-6}
0.003	2.5426×10^{-21}	1.2423×10^{-24}	4.0950×10^{-23}	5.2397×10^{-22}	3.9997×10^{-6}
0.004	5.1483×10^{-20}	2.2017×10^{-23}	6.2849×10^{-22}	1.0622×10^{-20}	8.4890×10^{-6}
0.01	7.1731×10^{-16}	2.0742×10^{-19}	3.7447×10^{-18}	1.4911×10^{-16}	9.2549×10^{-5}
0.02	8.9640×10^{-13}	2.0804×10^{-16}	2.6559×10^{-15}	1.8924×10^{-13}	5.5412×10^{-4}
0.03	5.3253×10^{-11}	1.1746×10^{-14}	1.2243×10^{-13}	1.1480×10^{-11}	1.5526×10^{-3}
0.04	8.8576×10^{-10}	2.0412×10^{-13}	1.8426×10^{-12}	1.9675×10^{-10}	3.1754×10^{-3}

Table 4 Comparison with the method in [19]. The absolute error in Example 5.2 is shown. .

n	$L_\infty - error$ in our method	$L_\infty - error$ in Ref [19]
4	1.6697×10^{-28}	4.01×10^{-4}
8	7.2304×10^{-27}	8.61×10^{-12}
12	7.2129×10^{-26}	3.17×10^{-15}
16	3.7649×10^{-25}	2.35×10^{-17}

The maximum absolute error in [19] with $n = 4$ is analyzed in which has result 4.01×10^{-4} while the maximum absolute error using our method which is presented in table 2 and table 3. The numerical results of this problem show that our method converges exponentially and is more accurate than the method [19] in table 4.3

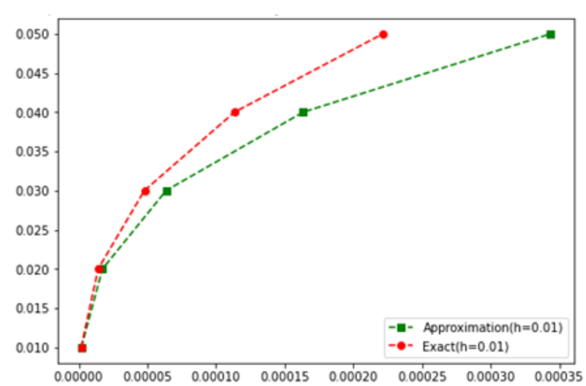

Figure 1 Approximate solution and exact solution for example 5.1, when$h = 0.01$.

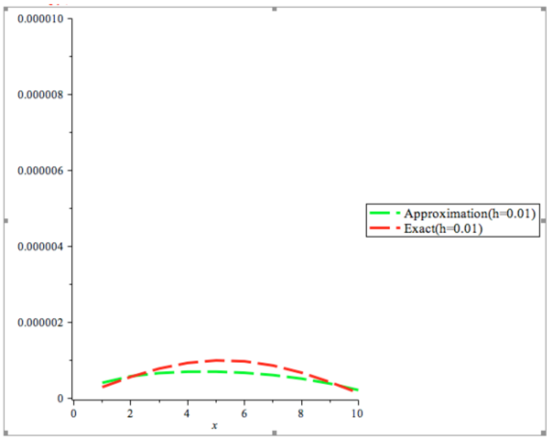

Figure 2 The maximal absolute error, when $h = 0.01$ for example 5.2.

5. Conclusion

In the paper, we have successfully introduced a scheme of spline function and constructed a polynomial spline numerical procedure based on the fractional order to solve fractional differential equations. All constant coefficients as fractional derivative type Riemann Liouville derivative, and the continuity condition, with fractional derivatives, so numerical accuracy and computational efficiency depend on the step size of h and this scheme spline method is easy and used to calculate the Riemann Liouville derivative. The two examples illustrate that a given method can successfully approximate the answer; In comparison to those methods already spline methods developed than the results show in the figures, on the other hand. The proposed method approximates the higher-order derivatives while concurrently approximating the solution of fractional initial value problem. A spline method is also developed based on the coefficient and the convergence of the approach.

6. References

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تحليل التقارب لدالة سبلاين المكعبة بدرجة كسرية وتطبيقات

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الملخص

في هذا البحث ، تم اقتراح مخطط سبلاين المكعب بدرجة كسرية وتحليله لمرتبة كسرية مع مشتقات ريمان ليوفيل متعددة الحدود (L-R). بالنسبة للمعادلات التفاضلية و التكاملية الكسرية ، نتعامل مع معادلات الاستمرارية الكسر ية ونحقق نظاما من المعادلات الجبرية الخطية باستخدام طريقة المصفوفة القائمة على دوال الاختبار الخطي متعدد التعريف.ان حدود تكامل ريمان-ليوفيل الكسري تم التعامل معها بواسطة تربيع الالتواء. طريقة مسائل القيمة الأبتدائية الكسرية لتقريب حل المعادلة الكسرية مع تقريب سبلاين لمشتق ريمان-ليوفيل. من أجل الحصول على طريقة منفصلة تماما ، تم استخدام تقريب سبلاين القياسي لفصل المشتق المكاني بشروط الاستمرارية المناسبة لطريقة المخطط بشرط أن يكون النموذج فريدا وموجودا لجميع الفاصل الزمني الذي يظهر في هذا المخطط للدالة وجميع المشتقات بمرتبة كسرية. تم إثبات تحليل التقارب بدقة من خلال طريقة سبلاين. بالإضافة إلى ذلك ، تم إثبات وجود وتفرد الحلول العددية للأنظمة الخطية بشكل صارم. تؤكد النتائج العددية التحليل النظري وتظهر فعالية الطريقة.