P- ISSN 1991-8941 E-ISSN 2706-6703
2023,(17), ( 2 ):338– 342
*Journal of University of Anbar for Pure Science (JUAPS)* Open Access

# A Further Generalization of the General Polynomial Transform and its Basic Characteristics and Applications.

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**ARTICLE INFO**
Received: 22 / 09 /2023
Accepted: 15 / 10 / 2023
Available online: 19 / 12 / 2023

**DOI: 10.37652/juaps.2023.143479.1140**

**Keywords:**
General Polynomial Transform,
AEM Transform,
differential equations.



**ABSTRACT**

In this article, we developed the general polynomial transform into a new transform (Ahmad-Emad-Murat transform ), which was expanded by writing a general formula of the Kernel function $K(x,t)$. Besides, we presented the essential characteristics and theorems of AEM transform and made new results. In addition, the efficiency of the proposed transform was verified by applying it to a set of important examples, the most important one is "Cauchy Euler problems". The main advantage of the proposed transform is getting a more generalized transform and making it easier to handle in solving differential equations with variable coefficients, reducing effort and time in the calculations. Hence, the polynomial integral transform and general polynomial transform that have been introduced during the last years are special transforms of the AEM transform.

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## Introduction:

Differential equations are one of the most interesting problems in applied mathematics. Because they have various applications in both engineering and science [3,5,10]. Several transformations have been presented to solve this problem, including the Fourier transform [8] and the Laplace transform. Recently, new transforms that depend mainly on the development of the Laplace transform have been discovered, including the Aboodh transform [2], Elzaki transform [6], complex SEE transform [4], Emad–Sara integral transform [9], Polynomial integral transform [1], General Polynomial transform [7], Al-Temimi transformation [5], N-transform [11], and other transforms. In this study, we developed General Polynomial transform by writing the kernel function in its general form. And with this, we get a more generalized transformation. In this paper, we will employ AEM Transform to solve differential equations with variable coefficients, as the following:

$$\beta_n t^n y^{(n)} + \beta_{n-1} t^{n-1} y^{(n-1)} + \cdots + \beta_1 t y' + \beta_0 y = \mu(t),$$

where $\beta_0, \beta_1, \cdots, \beta_n$ are constants, and $\mu(t)$ is known function.

## Definitions, properties, and theorems of AEM Transform :

**Definition 1**. [7] The General Polynomial Transform of $f(t)$ denoted by the operator $F$ is given by

$$I_s[f(t)] = \int_{t=1}^{\infty} t^{-(p(\alpha)+1)} f(t)\, dt = F\big(p(\alpha)\big).$$

Now, we will introduce the definition of Ahmad-Emad-Murat transform and its properties :

**Definition 2**. The Ahmad-Emad-Murat Transform of $f(t)$ denoted by the operator $E(H(\alpha), p(\alpha))$ is given by

$$AEM[f(t)] = H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)} f(t)\, dt = E\big(H(\alpha), p(\alpha)\big).$$

Where $H(\alpha), p(\alpha)$ are functions of parameter $\alpha$.

**Definition 3**. The inverse of Ahmad-Emad-Murat Transform of $E(p(\alpha), p(\alpha))$ denoted by $(AEM)^{-1}$ and

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defined as

$$(AEM)^{-1}\left[AEM[f(t)]\right]=f(t)$$
$$=\frac{1}{2\pi i}\int_{\delta-i\tau}^{\delta+i\tau}\frac{t^{p(\alpha)+1}}{H(\alpha)}E\big(H(\alpha),p(\alpha)\big)\,d\alpha.$$

In general $w=\delta+i\tau$ with $\delta$ and $\tau$ being real numbers, $i\in\mathbb{C}$. The integral converges when $R[\alpha]=\delta>0$ and $\delta<0$, $E\big(H(\alpha),p(\alpha)\big)=0$.

Some properties of AEM Transform are as follows:

- (Linearity of AEM Transform) If $f(t)=Cg(t)+Dh(t)$, then

$$AEM[Cg(t)\pm Dh(t)]$$
$$=H(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}(Cg(t)\pm Dh(t))\ dt,$$
$$=CH(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}g(t)\ dt\pm DH(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}h(t)\ dt,$$
$$=C\ AEM[g(t)]\pm D\ AEM[h(t)].$$

- If $f(t)=C, C$ is constant, then

$$AEM[C]=H(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}(C)\,dt$$
$$=CH(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}\ dt,$$
$$=CH(\alpha)\left[\frac{t^{-p(\alpha)}}{-p(\alpha)}\right]_1^{\infty}=\frac{CH(\alpha)}{p(\alpha)},p(\alpha)>0.$$

- If $f(t)=t^m$, then

$$AEM[t^m]=H(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}(t^m)\ dt$$
$$=H(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1-m)}\ dt$$
$$=H(\alpha)\left[\frac{t^{-p(\alpha)+m}}{-p(\alpha)+m}\right]_1^{\infty}=\frac{H(\alpha)}{p(\alpha)-m},p(\alpha)>m.$$

- If $f(t)=\ln t, t>1$ then

$$AEM[\ln t]=H(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}\ln t\ \ dt,$$
$$=H(\alpha)(\ln t\ \frac{t^{-p(\alpha)}}{-p(\alpha)}|_1^{\infty}-\int_{t=1}^{\infty}\frac{1}{t}\frac{t^{-p(\alpha)}}{-p(\alpha)}\ dt)$$
$$=\frac{H(\alpha)}{p(\alpha)}\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}\ dt=\frac{1}{p(\alpha)}.AEM[1]$$
$$=\frac{H(\alpha)}{\big(p(\alpha)\big)^2},p(\alpha)>0.$$

- If $f(t)=t^m\ln t, t>1, m\in\mathbb{R}$ then

$$AEM[t^m\ln t]=H(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}\,t^m\ln t\ \ dt,$$
$$=H(\alpha)(\ln t\ \frac{t^{-p(\alpha)+m}}{-p(\alpha)+m}|_1^{\infty}-\int_{t=1}^{\infty}\frac{1}{t}\frac{t^{-p(\alpha)+m}}{-p(\alpha)+m}\ dt)$$
$$=\frac{H(\alpha)}{p(\alpha)-m}\int_{t=1}^{\infty}t^{-(p(\alpha)+1)+m}\ dt$$
$$=\frac{1}{p(\alpha)-m}.AEM[t^m]$$
$$=\frac{H(\alpha)}{(p(\alpha)-m)^2},p(\alpha)>m.$$

- If $f(t)=\sin(a\ln(t)), t>1, a\in\mathbb{R}$, then

$$AEM[\sin(a\ln(t))]$$
$$=H(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}\sin(a\ln(t))\ \ dt,$$

by integrating by parts twice, we get

$$AEM[\sin(a\ln(t))]=\frac{a\,H(\alpha)}{\big(p(\alpha)\big)^2+a^2}.$$

- If $f(t)=\cos(a\ln(t)), t>1, a\in\mathbb{R}$, then

$$AEM[\cos(a\ln(t))]$$
$$=H(\alpha)\int_{t=1}^{\infty}t^{-(p(\alpha)+1)}\cos(a\ln(t))\ \ dt,$$

by integrating by parts twice, we get

$$AEM[\cos(a\ln(t))]=\frac{H(\alpha)\,p(\alpha)}{\big(p(\alpha)\big)^2+a^2}.$$

- If $f(t)=\sinh(a\ln(t)), t>1, a\in\mathbb{R}$, then

$$AEM[\sinh(a\ln(t))]=\frac{a\,H(\alpha)}{\big(p(\alpha)\big)^2-a^2},|p(\alpha)|>a.$$

- If $f(t)=\cosh(a\ln(t)), t>1, a\in\mathbb{R}$, then

$$AEM[\cosh(a\ln(t))] = \frac{H(\alpha)\, p(\alpha)}{\left(p(\alpha)\right)^2 - a^2}, |p(\alpha)| > a.$$

**Theorem1.** If $m \in \mathbb{R}$ and $AEM[f(t)] = E\big(H(\alpha), p(\alpha)\big)$, then $AEM[t^m f(t)] = E(H(\alpha), p(\alpha) - m)$.

$$AEM[t^m f(t)] = H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)} t^m f(t)\, dt,$$

$$= H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1-m)}\, f(t) dt$$

$$= E(H(\alpha), p(\alpha) - m).$$

**AEM Transform of derivatives of $f(t)$ :**

**1.**

$$AEM[tf'(t)] = H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)} t f'(t)\ dt$$

$$= H(\alpha) \int_{t=1}^{\infty} t^{-p(\alpha)} f'(t)\ dt,$$

$$= H(\alpha)(t^{-p(\alpha)}\, f(t)|_1^{\infty} + p(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)}\, f(t)\ dt\,)$$

$$= -f(1)\, H(\alpha) + p(\alpha) E\big(H(\alpha), p(\alpha)\big).$$

**2.**

$$AEM[t^2 f''(t)] = H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)} t^2 f''(t)\ dt$$

$$= H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)-1)} f''(t)\ dt,$$

$$= H(\alpha)(t^{-(p(\alpha)-1)}\, f'(t)|_1^{\infty} + (p(\alpha) - 1) \int_{t=1}^{\infty} t^{-p(\alpha)}\, f'(t)\ dt)$$

$$= H(\alpha)(-f'(1) + (p(\alpha) - 1)[t^{-p(\alpha)}\, f(t)|_1^{\infty} + p(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)}\, f(t)\ dt])$$

$$= -H(\alpha) f'(1) - H(\alpha) f(1)(p(\alpha) - 1) + p(\alpha)(p(\alpha) - 1) E\big(H(\alpha), p(\alpha)\big).$$

**Theorem 2**. Let $f(t)$ be continuous on $(1, \infty)$ and if $f'(t), f''(t), \cdots, f^{(n)}(t)$ are exists, then

$$AEM\left[t^m f^{(m)}(t)\right] = -H(\alpha) f^{(m-1)}(1) - H(\alpha)\big(p(\alpha) - (m-1)\big) f^{(m-2)}(1) - H(\alpha)\big(p(\alpha) - (m-1)\big)\big(p(\alpha) - (m-2)\big) f^{(m-3)}(1) - \cdots - H(\alpha)\big(p(\alpha) - (m-1)\big)\big(p(\alpha) - (m-2)\big)\big(p(\alpha) - (m-3)\big) \cdots (p(\alpha) - 1) f(1) + \frac{p(\alpha)!}{(p(\alpha)-m)!} E\big(H(\alpha), p(\alpha)\big).$$

**Proof**. Let we do with mathematical induction

i. Is true for $m = 1$

$$AEM[tf'(t)] = H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)} t f'(t)\ dt$$

$$= H(\alpha) \int_{t=1}^{\infty} t^{-p(\alpha)} f'(t)\ dt,$$

$$= H(\alpha)(t^{-p(\alpha)}\, f(t)|_1^{\infty} + p(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)}\, f(t)\ dt\,)$$

$$= H(\alpha)(-f(1) + p(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)}\, f(t)\ dt)$$

$$= -f(1)\, H(\alpha) + p(\alpha) AEM[f(t)]$$

$$= -f(1)\, H(\alpha) + p(\alpha) E\big(H(\alpha), p(\alpha)\big).$$

for $m = 1$ is true.

ii. Let true for $m$ that is

$$AEM\left[t^m f^{(m)}(t)\right] = -H(\alpha) f^{(m-1)}(1) - H(\alpha)\big(p(\alpha) - (m-1)\big) f^{(m-2)}(1) - H(\alpha)\big(p(\alpha) - (m-1)\big)\big(p(\alpha) - (m-2)\big) f^{(m-3)}(1) - \cdots - H(\alpha)\big(p(\alpha) - (m-1)\big)\big(p(\alpha) - (m-2)\big)\big(p(\alpha) - (m-3)\big) \cdots (p(\alpha) - 1) f(1) + \frac{p(\alpha)!}{(p(\alpha) - m)!} E\big(H(\alpha), p(\alpha)\big).$$

iii. We must show true for $m + 1$

$$AEM\left[t^{m+1} f^{(m+1)}(t)\right]$$

$$= H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)+1)} t^{m+1} f^{(m+1)}(t)\, dt$$

$$- H(\alpha) f^{(m)}(1)$$

$$= H(\alpha) \int_{t=1}^{\infty} t^{-(p(\alpha)-m)} f^{(m+1)}(t)\, dt =$$

$$= H(\alpha)(t^{-(p(\alpha)-m)} f^{(m)}(t)|_1^{\infty} + (p(\alpha) - m) \int_{t=1}^{\infty} t^{-(p(\alpha)-m+1)}\, f^{(m)}(t)\ dt\ )$$

$$= -H(\alpha) f^{(m)}(1) + (p(\alpha) - m) AEM\left[t^m f^{(m)}(t)\right]$$

$$= -H(\alpha) f^{(m)}(1) + (p(\alpha) - m)\Big[-H(\alpha) f^{(m-1)}(1) - H(\alpha)\big(p(\alpha) - (m-1)\big) f^{(m-2)}(1) - H(\alpha)\big(p(\alpha) - (m-1)\big)\big(p(\alpha) - (m-2)\big) f^{(m-3)}(1) - \cdots - H(\alpha)\big(p(\alpha) - (m-1)\big)\big(p(\alpha) - (m-2)\big)\big(p(\alpha) - (m-3)\big) \cdots (p(\alpha) - 1) f(1) + \frac{p(\alpha)!}{(p(\alpha) - m)!} E\big(H(\alpha), p(\alpha)\big)\Big] =$$

$$+\Big[-H(\alpha)(p(\alpha) - m) f^{(m-1)}(1) - H(\alpha)(p(\alpha) - m)\big(p(\alpha) - (m-1)\big) f^{(m-2)}(1) - H(\alpha)(p(\alpha) - m)\big(p(\alpha) - (m-1)\big)\big(p(\alpha) - (m-2)\big) f^{(m-3)}(1) - \cdots - H(\alpha)(p(\alpha) - m)\big(p(\alpha) - (m-1)\big)\big(p(\alpha) - (m-2)\big)\big(p(\alpha) - (m-3)\big) \cdots (p(\alpha) - 1) f(1) + \frac{p(\alpha)!}{(p(\alpha) - (m+1))!} E\big(H(\alpha), p(\alpha)\big)\Big]$$

Thus, for $m + 1$ is true.

## Examples of Applying AEM Transform on Differential Equations with Variable Coefficients:

In this section two examples are given and exact solution is found using our new transformation.

**Example 1.** [5] consider the following equation

$$t^2 y'' + 6ty' + 6y = \frac{1}{t^2}, (1)$$

with ICs $y'(1) = 2, y(1) = -4$.

By using the AEM Transform into equation (1), we get:

$$AEM[t^2 y''] + 6\, AEM[ty'] + 6\, AEM[y] = AEM\left[\frac{1}{t^2}\right]. (2)$$

Equation (2) can be written in the form

$$-H(\alpha) y'(1) - H(\alpha) y(1)(p(\alpha) - 1) + p(\alpha)(p(\alpha) - 1) E\big(H(\alpha), p(\alpha)\big) - 6H(\alpha) y(1)$$

$$+6\, p(\alpha) E\big(H(\alpha), p(\alpha)\big) + 6\, E\big(H(\alpha), p(\alpha)\big) = \frac{H(\alpha)}{p(\alpha)+2}$$

and by applying the initial condition and simplify equation, we obtain :

$$E\big(H(\alpha), p(\alpha)\big) = H(\alpha)\, \frac{2(p(\alpha))^2 + 10\, p(\alpha) + 13}{(p(\alpha) + 2)^2 (p(\alpha) + 3)}$$

By using the partial fraction of the last equation, we have

$$E\big(H(\alpha), p(\alpha)\big) = \frac{H(\alpha)}{p(\alpha) + 2} + \frac{H(\alpha)}{(p(\alpha) + 2)^2} + \frac{H(\alpha)}{p(\alpha) + 3}. (3)$$

By using the inverse AEM transform into equation (3), we get the solution of equation (1), that is,

$y(t) = t^{-2} + t^{-2} \ln t + t^{-3}$.

**Example 2.** consider the following equation

$$ty' + 2y = \cos(2 \ln t)\, , (4)$$

with IC $y(1) = 1$.

By using the AEM Transform into equation (4), we get:

$$AEM[ty'] + 2\, AEM[y] = AEM[\cos(2 \ln t)]. (5)$$

Equation (5) can be written in the form

$$-H(\alpha) y(1) + p(\alpha) E\big(H(\alpha), p(\alpha)\big) + 2\, E\big(H(\alpha), p(\alpha)\big) = \frac{H(\alpha)\, p(\alpha)}{(p(\alpha))^2 + 4}$$

and by applying the initial condition and simplify equation, we obtain:

$$E\big(H(\alpha), p(\alpha)\big) = \frac{H(\alpha)}{p(\alpha) + 2} + \frac{H(\alpha)\, p(\alpha)}{(p(\alpha) + 2)(p(\alpha))^2 + 4}$$

By using the partial fraction of the last equation, we have

$$E\big(H(\alpha), p(\alpha)\big) = \frac{3}{4}\, \frac{H(\alpha)}{p(\alpha) + 2} + \frac{1}{4}\, \frac{H(\alpha)\, p(\alpha)}{(p(\alpha))^2 + 4} + \frac{1}{2}\, \frac{H(\alpha)}{(p(\alpha))^2 + 4}. (6)$$

By using the inverse AEM transform into equation (6), we get the solution of equation (4), that is,

$$y(t) = \frac{3}{4}t^{-2} + \frac{1}{4}\cos(2\ln t) + \frac{1}{4}\sin(2\ln t).$$

## Conclusion

In this study, a new type of transformation (AEM transform) was constructed and applied to differential equations with variable coefficients. And then we used it to solve a class of “Euler equation”. The main advantage of the proposed transform for solving this equation is to find solutions without dealing with complex calculations, and to get a more generalized transform from a general polynomial transform. Furthermore, from the definition of the AEM transform, we can construct many new integral transformations by selecting new formulas for $H(\alpha)$ and $p(\alpha)$. In future studies the proposed transformation can be used to solve differential difference equations.

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# المزيد من التعميم لتحويل كثير الحدود العام وخصائصه الأساسية وتطبيقاته

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**الخلاصة:**
في هذه المقالة، قمنا بتطوير تحويل كثير الحدود العام إلى تحويل جديد (تحويل أحمد-عماد-مراد)، والذي تم توسيعه عن طريق كتابة صيغة عامة لدالة النواة K(x,t). علاوة على ذلك، قدمنا الخصائص والنظريات الأساسية لتحويل AEM وحققنا نتائج جديدة. بالإضافة إلى ذلك تم التحقق من كفاءة التحويل المقترح من خلال تطبيقه على مجموعة من الأمثلة المهمة أهمها "مسائل كوشي أويلر". الميزة الرئيسية للتحويل المقترح هو الحصول على تحويل أكثر عمومية ويسهل التعامل في حل المعادلات التفاضلية ذات المعاملات المتغيرة، مما يقلل الجهد والوقت في العمليات الحسابية. لذلك فإن تحويل كثير الحدود وتحويل كثير الحدود العام الذي تم تقديمه خلال السنوات الأخيرة هما تحويلات خاصة من تحويل AEM.

**الكلمات المفتاحية**: تحويل كثير الحدود العام، تحويل أحمد، معادلات تفاضلية.