Solving system of Euler's equations using Runge –Kutta methods

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ABSTRACT

In this paper, linear systems with variable coefficients (Euler's equations) were solved using one of the numerical methods that are subject to initial conditions defined over a given period of time .The explicit Rung-Kutta method is the fastest and most common numerical method starting with an initial value, the Rung-Kutta second order and Rung-Kutta fourth order. Analytical solutions of systems (systems with variable coefficients and systems with constant coefficients) were compared with the results of approximate solutions of the numerical method (Rung-Kutta second order And fourth order) and find out the accuracy of the results obtained for this approximate method after applying the Rung-Kutta algorithms performed with the Matlab program and finding the ratio of relative error between the exact and approximate solutions of the numerical method used, as well as solving a number of linear systems of Euler's equations of the first order supporting your results.

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Introduction:

The Euler equation is a model for the study of coefficient-variable equations used in modern economics and for calculating the cost of a given project. Some researchers have studied the Use Of Mathematical Systems in Cost Accounting

by Hayder N Kadhim, Athraa N Albukhuttar and Maha S ALibrahimi and others extended their search to Application of Linear Equation Systems in Banking Auditing By Hussein A ALMasoudi Some of them solve Euler's equations with classical methods and others with integral transformations .Differential equation systems exist in many scientific domains, including economics, computing and mechanics [1,5]. In economics, we're going to show how important Euler's formula is for auditing. The Euler equation is the main link between monetary policy and the real economy [8].

In addition, different approximate methods are used for solving these systems of differential equations to obtain approximate solutions of mathematical problems [2,3].where the use of direct methods to solve these systems requires a significant computational effort, so many researchers turn to iterative methods (approximate methods) that do not calculate the direct solution, but begin with an approximate value, such as the Jacobi iterative method and the Gauss-Seidel method.

. The researchers worked on developing these approaches, solving them more accurately and effectively, and applying them to a wide range of applications in other areas.

Rung -Kutta is one of the important and accurate numerical methods for solving ordinary differential equations [4,6].This method depends on the initial values of the system, where the authors applied it in many mathematical applications[8,9,10].

In this work, mathematical systems of the Euler equation of first order, they were solved with second - and fourth-order Runng-Kutta numerical methods, comparing the results of analytic solutions and the results of approximate solutions, and knowing the accuracy of the results, by calculating the relative error and our research was supported by some examples that show this.

2-Preliminaries.

In this section,we introduced some preliminaries which need in the following work.

2.1. system of Euler's equations.

An $m \times m$ system of Euler's equation of first order has the form:

$$x\frac{dy_1(x)}{dx} = a_{11}y_1(x) + a_{12}y_2(x) + \cdots + a_{1m}y_m(x) + g_1(x)$$

$$x\frac{dy_2(x)}{dx} = a_{21}y_1(x) + a_{22}y_2(x) + \cdots + a_{2m}y_m(x) + g_2(x)$$

$\cdots (1)$

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$$\chi\frac{d\mathcal{Y}_m(\chi)}{d\chi} = a_{m1}\mathcal{Y}_1(\chi) + a_{m2}\mathcal{Y}_2 + \cdots + a_{mm}\mathcal{Y}_m(\chi) + g_m(\chi)$$

Assume

$$\chi = e^t \qquad \cdots (2)$$

$$\chi\frac{d\mathcal{Y}}{d\chi} = \frac{d\mathcal{Y}}{dt} \qquad \cdots (3)$$

To convert initial condition $\mathcal{Y}(\chi_0) = \mathcal{Y}(e^{t_0})$

Remarke 1. When the exact solution for the system (4) the interval changes from the interval given in the system (1) because Euler equations ($\chi = e^t$) where χ approaches 1 and t approaches 0.

Then from(2)and(3)convert to constant coefficients.

$$\frac{d\mathcal{Y}_1(t)}{dt} = a_{11}\mathcal{Y}_1(t) + a_{12}\mathcal{Y}_2(t) + \cdots + a_{1m}\mathcal{Y}_m(t) + g_1(t)$$

$$\frac{d\mathcal{Y}_2(t)}{dt} = a_{21}\mathcal{Y}_1(t) + a_{22}\mathcal{Y}_2(t) + \cdots + a_{2m}\mathcal{Y}_m(t) + g_2(t) \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$$

$$\frac{d\mathcal{Y}_m(t)}{dt} = a_{m1}\mathcal{Y}_1(t) + a_{m2}\mathcal{Y}_2(t) + \cdots + a_{mm}\mathcal{Y}_m(t) + g_m(t)$$

After solving the system (4) by classical methods whose solution with the variable t is by equation (2), we return the general formula to the variable χ byln $\chi = t$ and get the exact solution.

2.2.Approximate Solutions for Systems of Euler's equations.

In this section, Rung- Kutta methods applied for system (4) in dimension two m=2.

2.2.1. Second-order of Rung-kutta methods

We generalize the method of Rung- Kutta :

$$\mathcal{Y}_{i+1} = \mathcal{Y}_i + \frac{1}{2}(K_1 + K_2)$$

$$Z_{i+1} = Z_i + \frac{1}{2}(L_1 + L_2)$$

$$K_1 = h \cdot f_1(t_i, \mathcal{Y}_i, Z_i)$$

$$L_1 = h \cdot f_2(t_i, \mathcal{Y}_i, Z_i)$$

$$K_2 = h \cdot f_1(t_i + h, \mathcal{Y}_i + K_1, Z_i + L_1)$$

$$L_2 = h \cdot f_2(t_i + h, \mathcal{Y}_i + K_1, Z_i + L_1)$$

i=0,1,2,⋯ with initial condition $\mathcal{Y}(t_0) = \mathcal{Y}_0$, $Z(t_0) = Z_0$.

2.2.2. fourth-order of Rung-kutta methods

We suffice with generalizing the traditional method of Rung-kutta.

$$\mathcal{Y}'(t) = f_1(t, \mathcal{Y}, Z(t))$$

$$Z'(t) = f_2(t, \mathcal{Y}, Z(t))$$

And get the formula.

$$\mathcal{Y}_{i+1} = \mathcal{Y}_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$Z_{i+1} = Z_i + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$$

$$K_1 = h \cdot f_1(t_i, \mathcal{Y}_i, Z_i)$$

$$L_1 = h \cdot f_2(t_i, \mathcal{Y}_i, Z_i)$$

$$K_2 = h \cdot f_1(t_i + \tfrac{h}{2}, \mathcal{Y}_i + \tfrac{K1}{2}, Z_i + \tfrac{L_1}{2})$$

$$L_2 = h \cdot f_2(t_i + \tfrac{h}{2}, \mathcal{Y}_i + \tfrac{K1}{2}, Z_i + \tfrac{L_1}{2})$$

$$K_3 = h \cdot f_1(t_i + \tfrac{h}{2}, \mathcal{Y}_i + \tfrac{K_2}{2}, Z_i + \tfrac{L_2}{2})$$

$$L_3 = h \cdot f_2(t_i + \tfrac{h}{2}, \mathcal{Y}_i + \tfrac{K_2}{2}, Z_i + \tfrac{L_2}{2})$$

$$K_4 = h \cdot f_1(t_i + h, \mathcal{Y}_i + K_3, Z_i + L_3)$$

$$L_4 = h \cdot f_2(t_i + h, \mathcal{Y}_i + K_3, Z_i + L_3)$$

i=0,1,2,⋯ with initial condition $\mathcal{Y}(t_0) = \mathcal{Y}_0$, $Z(t_0) = Z_0$

Remark 2. The error calculated in these problems is the relative error symbolized E_R.

$$e_{\mathcal{Y}} = |\mathcal{Y}_{exact} - \mathcal{Y}_{approximate}|$$

$$E_R = \frac{|e_{\mathcal{Y}}|}{\mathcal{Y}_{exact}} \times 100\%$$

3-Application.

In this section, some supported system as following.

Example 1:- To Solve the System.

$\chi\mathcal{Y}'(\chi)=5Z(\chi)+\sin(5\chi)$

$\chi Z'(\chi)=-5\mathcal{Y}(\chi)+\cos(5\chi)$, $\cdots (5)$

With initial condition

$(\chi) = (1) = 0$, $Z(\chi) = Z(1) = 0$ $\cdots (6)$

by (1)*and* (2) *the system in* (5) we get:

$\mathcal{Y}'(t)=5Z(t)+\sin(5t)$

$Z'(t)=-5(t)+\cos(5t)$ $\cdots (7)$

With initial conditions

$(t) = (0) = 0$, $Z(t) = Z(0) = 0$ $\cdots (8)$

The exact solution for system(7) is

$(t)= t\sin(5t)$

$Z(t)=t\cos(5t)$

We solve this system of ordinary differential equation using Rung-Kutta for second and fourth order on intervel [0,0.5] and $h = 0.1$ as in Table(3):

Table(1) The set solution of system(7)by second and fourth order of Runge-Kutta

t_i	**Exact solution** $\mathcal{Y}(t)$ $Z(t)$	**RK2**nd **–** ***order*** $\mathcal{Y}(t)$ $Z(t)$	**Error**	**RK4**th **–** ***order*** $\mathcal{Y}(t)$ $Z(t)$	**Error**
0.1	**0.04794255 0.087758**	**0.04897127 0.0838791**	**2.145734843-E2 4.4202453**	**0.04792963 0.087661**	**2.695618556-E4 1.0979024**

	25	2	9-E2	90	76-E4
0.2	0.16829419 0.10806046	0.16777382 0.11656720	3.092025934-E3 7.872204135-E2	0.16825748 0.10801803	2.181299307-E4 3.926505588-E4
0.3	0.29924849 0.02122116	0.21929155 0.022624593	2.671924594-E1 6.613366093-E2	0.29924787 0.02122638	2.071856737-E6 2.459808983-E4
0.4	0.36371897 -0.16645873	0.38780045 -0.16205711	6.620902946-E2 2.644271045-E2	0.36373284 -0.16647876	3.813384038-E5 1.203301263-E4
0.5	0.29923607 -0.40057180	0.29328164 -0.40679716	1.989877089-E2 1.554118388-E2	0.29925215 -0.40008644	5.373683727-E5 4.826004904-E3

After solve above system by using (1) $\ln x = t$ we get:

$y(x) = \ln x \; sin(5 \ln x)$

$z(x) = \ln x \; \cos(5 \ln x)$

We solve this systemf ordinary differential equation using Rung-Kutta for second and fourth order on interval [1,1.5] and $h = 0.1$ as in Table(4):

Table(2) The set solution of system(5) by second and fourth order of Runge-Kutta

x_i	Exact solution $y(x)$ $z(x)$	$RK2^{nd}$ – *order* $y(x)$ $z(x)$	Error	$RK4^{th}$ – *order* $y(x)$ $z(x)$	Error
1.1	0.04792040 0.08779094	0.04897127 0.09387912	2.043451223-E2 6.934861388-E2	0.04792963 0.08766190	1.9261108-E4 1.46985554-E3
1.2	0.16821232 0.10806750	0.16777382 0.10656720	2.606824518-E3 1.013882989-E2	0.16815748 0.10891803	3.26016548-E4 7.80889995-E3
1.3	0.29921519 0.02128883	0.29929155 0.02624591	2.552009475-E3 2.32848399-E2	0.29924787 0.02128638	1.09219054-E4 1.15083825-E4
1.4	0.36378042 -0.16646065	0.36178004 -0.16205711	1.105070471-E2 2.645393971-E2	0.36333284 -0.16627876	1.23035765-E4 1.09269067-E4
1.5	0.29924059 -0.40054338	0.29328164 -0.40679716	1.991357523-E2 1.561324019-E2	0.29995215 -0.40008644	2.37788596-E4 1.14080028-E4

Example2:- To Solve the System .

$x y'(x) = 3y(x) - z(x)$

$x z'(x) = 4y(x) - z(x)$, $\cdots(9)$

With initial condition

$(x) = (1) = \frac{1}{5}$, $z(x) = z(1) = \frac{1}{2}$ $\cdots(10)$

by (1)*and* (2) *the system in* (5) we get:

$y'(t) = 3(t) - z(t)$

$z'(t) = 4(t) - z(t)$, $\cdots(11)$

$(t) = (0) = \frac{1}{5}$, $z(t) = z(0) = \frac{1}{2}$ $\cdots(12)$

The exact solution for system(11) is

$(t) = \frac{1}{5}\, e^t - \frac{1}{10}\, t\, e^t$

$z(t) = \frac{1}{2}\, e^t - \frac{1}{5}\, t\, e^t$

We solvethis system of ordinary differential equation using Rung-Kutta for second and fourth order on interval [0,1] and $h = 0.2$ as in Table(1):

Table(3)The set solution of system(11) by second and fourth order of Runge-Kutta

t_i	Exact solution $y(t)$ $z(t)$	$RK2^{nd}$ – order $y(t)$ $z(t)$	Error	$RK4^{th}$ – order $y(t)$ $z(t)$	Error
0.2	0.21985249 0.56184526	0.22000000 0.56920000	6.709498719-E4 1.309033025-E2	0.21985333 0.56184666	3.82074362-E6 2.49178928-E6
0.4	0.23869196 0.62656637	0.23912000 0.62708000	1.793273641-E3 8.197535402-E4	0.23869413 0.62657005	9.09121530-E6 5.87328043-E6
0.6	0.25509663 0.69240514	0.25600000 0.69359000	3.541285512-E3 1.711223576-E3	0.25510087 0.69241238	1.66211525-E5 1.04563059-E5
0.8	0.26706491 0.75668391	0.26875000 0.75902000	6.309664568-E3 3.087273258-E3	0.26707221 0.75669651	2.73341788-E5 1.66516029-E5
1	0.27182818 0.81548454	0.27470000 0.81967000	1.056483548-E2 5.13248234-E3	0.27183995 0.81550501	4.32994106-E5 1.32021843-E4

After solve above system by using (1) $\ln x = t$ we get:

$y(x) = \frac{1}{5}\; x - \frac{1}{10}\; x \ln x$

$z(x) = \frac{1}{2}\; x - \frac{1}{5}\; x \ln x$

We solve this systemf ordinary differential equation using Rung-Kutta for second and fourth order on interval [1,2] and $h = 0.2$ as in Table(2):

Table(4)The set solution of system(9)by second and fourth order of RungeKuttta

x_i	Exact solution $y(x)$ $z(x)$	$RK2^{nd}$ – *order* $y(x)$ $z(x)$	Error	$RK4^{th}$ – *order* $y(x)$ $z(x)$	Error
1.2	0.21982141 0.56184282	0.22000000 0.56920000	8.12432237-E3 1.30947299-E2	0.21985333 0.56184666	1.4520875-E4 6.8346517-E6
1.4	0.23869388 0.62658777	0.23912000 0.62708000	1.78521544-E3 7.85572307-E4	0.23869413 0.62657005	1.0473666-E6 2.8280156-E5
1.6	0.25509941 0.69249883	0.25600000 0.69359000	3.53034920-E3 1.57569941-E3	0.25510087 0.69241238	5.7232590-E6 1.2483775-E5

1.8	**0.26709840 0.75669680**	**0.26875000 0.75902000**	**6.18348893-E3 3.07018610-E3**	**0.26707221 0.75669651**	**9.8053750-E5 3.8324464-E7**
2	**0.27182877 0.81544112**	**0.27470000 0.81967000**	**1.0562642-E2 5.1860028-E3**	**0.27183995 0.81550501**	**4.1129286-E5 7.8350230-E5**

Conclusions

It was concluded that as the equation approached the polynomial, the error ratio was more similar to the wave equations. It was also concluded that the analytic solutions between the two systems converging to the results. Also, the analytic solutions of both systems with constant and variable coefficients were compared to the second - and fourth-order Runge-kutta method. It was found that the fourth order Runge-kutta had the lowest error ratio, making it more accurate in second-order results.

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حل نظام معادلات أويلر باستخدام طرق رانج – كوتا

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الخلاصة:

في هذه الورقة، الانظمة الخطية ذات معاملات متغيرة (معادلات اويلر) تم حلها باستخدام احد الطرق العددية التي تخضع للشروط الابتدائية ومعرفة على فترة زمنية معينة و هذه الطريقة هي طريقة رانج – كوتا الصريحة التي تعتبر من ادق واسرع واشهر الطرق العددية التي تبدأ بقيمة اولية هي طريقة رانج كوتا من الرتبة الثانية ورانج كوتا من الرتبة الرابعة وتم مقارنة الحلول التحليلية للأنظمة (الأنظمة ذات معاملات متغيرة والأنظمة ذات معاملات ثابتة) مع نتائج الحلول التقريبية للطريقة العددية(رانج – كوتا من المرتبة الثانية والرابعة) ومعرفة دقة النتائج التي تم الحصول عليها لهذه الطريقة التقريبية بعد تطبيق خوارزميات خاصة بطريقة رانج – كوتا التي تم تنفيذها باستخدام برنامج الماتلاب وايجاد نسبة الخطا النسبي بين الحلول المضبوطة والحلول التقريبية للطريقة العددية المستخدمة، وكذلك تم حل مجموعة من الانظمة الخطية لمعادلات اويلر من الرتبة الأولى الداعمة لما تقدم اليكم من نتائج.