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An Initial-Boundary Value Problem with New Technique of Piecewise Uniform Mesh

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ABSTRACT

The objective of my research is to establish facts and determine their significance. A new ε −convergent piecewise uniform mesh has been produced, by deriving a hybrid technique to find out the extent of subdomains (τ) of the singular boundary layers that occur when solving some of the differential equation problems numerically, where ε, is set to multiply terms covering the highest derivatives in the differential equation, in which determinant is zero, these boundary layers are adjacent to the boundary of the domain, where the solution yields a very deep gradient. The mesh has been used with the difference scheme function code in the MATLAB program; specifically, PDEPE that is solving initial-boundary value problems pertained parabolic-elliptic PDEs. It was applied to solve multiple examples then comparing the maximum error of the solutions with its counterpart "uniform mesh" and proving its superiority. Results, solutions, and comparisons were exposed with concise explanatory MATLAB plots manifested in some necessary tables for comparative studies.

Introduction

Boundary layers can be worked out in the solution of singularly disordered problems whereby the singular perturbation parameter ε, is set to multiply terms covering the highest derivatives in the differential equation, in which the limit is zero. These boundary layers are adjacent to the boundary of the domain, where the solution yields a very deep gradient. A boundary layer to either regular or parabolic type may arise from any angle of the domain. When the features of the reduced equation correspond to $\varepsilon = 0$, then it is determined to be a parabolic type whereas the characteristics of the reduced equation are not parallel to the boundary layer near the corner is called corner type numerical methods using standards finite difference operators on uniform meshes are not appropriate for solving problems with boundary layers due to the formed deep gradients. In addition, convergence analysis lies in the maximum norm rather than in an averaged norm, so that the singular components can be detected. These considerations lead to the concept of an ε-uniform method.

That is a numerical method for solving singularly distorted problems with an error estimate in the maximum norm that relies on the size of the singular perturbation constant ε. When the solution is a regular boundary layer, it is often possible to obtain an ε-uniform method by setting up an adequately fitted finite difference operator on a uniform mesh. Yet, the approach is not possible when a parabolic boundary layer is found. This negative result cause first demonstrated Shishkin is present by constructing an appropriate However, this approach is not possible if a parabolic boundary layer is present. This negative result was first proved in Shishkin[1], The ultimate objective of this paper is to prove in-depth, the positive conclusion for linear parabolic problems with parabolic boundary layers. An ε-uniform method can be composed by using a standard finite difference operator on an adequately fitted piecewise uniform mesh concentrating in the boundary layers[2].

Among the most basic parabolic partial differential equation, is the diffusion equation that is manifested in one-space dimensions,

$$\varepsilon \frac{\partial y}{\partial x} = \frac{\partial^2 y}{\partial x^2} \qquad (1)$$

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Thus, the sub-equation

$$\varepsilon \frac{\partial y}{\partial x} = \nabla^2 u \qquad (2)$$

Is inherent to two or more space dimensions whereby ε some given constant [3]. The study of heat problems dominated research and experiments in 18th century when a large number of researches paper have been published on numerical methods for heat problems. However, papers comparing these methods are usually restricted to the analysis of stability within the schemes used. The parabolic type of PDE, which is a diffusion equation used to be known as heat equation typically contains layer means and narrow regions where the solution changes rapidly [4]. In this paper, a robust (ε-convergent) type numerical method is derived to solve problems of type singularly diffusion equation (i.e. when $0 < \varepsilon < 1$) in equation (1) obtaining the numerical method via MATLAB code PDEPE as a basis for deriving the new method. Whereas the numerical method with already exists code in the MATLAB program namely PDEPE is used to solve parabolic and elliptic partial differential equations problems in variables x and y, of the form:

$$c\left(x, y, u, \frac{\partial u}{\partial x}\right) \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}\left(f\left(x, y, u, \frac{\partial u}{\partial x}\right)\right) + s\left(x, y, u, \frac{\partial u}{\partial x}\right) \qquad (3.a)$$

The PDEs hold for $y_0 \le y \le y_f$ and $a \le x \le b$. The interval $[a, b]$ must be finite. In Equation (1), $f\left(x, y, u, \frac{\partial u}{\partial x}\right)$ is a flux term and $s\left(x, y, u, \frac{\partial u}{\partial x}\right)$ is a source term. The coupling of the partial derivatives with respect to y is restricted to multiplication by a diagonal matrix $c(x, y, u, \frac{\partial u}{\partial x})$. The diagonal elements of this matrix are either identically zero or positive. An element that is identically zero corresponds to an elliptic equation, otherwise corresponds to a parabolic equation. There must be at least one parabolic equation an element of c that corresponds to a parabolic equation can vanish at isolated values of x if those values of x are mesh points. Discontinuities in c and/or s due to material interfaces are permitted provided that a mesh point is placed at each interface. For $y = y_0$ and all x, the solution constituents fulfill the initial conditions of the form:

$$u(x, y_0) = u_0(x) \qquad (3.b)$$

For all y and either $x = a$ or $x = b$, the solution constituents fulfill a mixed Neumann with Dirichlet boundary conditions, of the form

$$p(x, y, u) + q(x, y) f\left(x, y, u, \frac{\partial u}{\partial x}\right) = 0 \qquad (3.c)$$

[5] [6] [7] [8] [9] [10] [11]

Definition

A strictly-assembled monotone function $\varphi: [0, 1] \rightarrow [0, 1]$ that maps a uniform mesh $t_i = i/N$, $i = 0, \ldots, N$, onto a layer-adjusted mesh by

$$x_i = \varphi(t_i), \qquad i = 0, \ldots, N,$$

is called a mesh generating function [خطأ! الإشارة المرجعية غير معرّفة.].

Bakhvalov Meshes

Bakhvalov is attributed to the Russian mathematician Nikolai Sergeevich Bakhvalov (1934-2005). Bakhbalov's theory is based on the initial idea of building a local uniform mesh with constant length of the step size closed to $x = 0$, on the curve, and then projecting the equal gradations on the x-axis using the (scaled) boundary layer function. Thus, grid points x_i near $x = 0$ are defined by,

$$q\left(1 - e^{-\beta x_i/\sigma\varepsilon}\right) = t_i = \frac{i}{N} \text{ for } i = 0,1, \ldots,$$

Whereby the scaling parameters $q \in (0, 1)$ and $\sigma > 0$ are user selected: q is roughly the portion of mesh points used to resolve the layer, while σ determines the grading of the mesh inside the layer. The part that containing the layer will have a local uniform mesh on the x-axis such that its length on the x-axis is equal to the transition point τ, as for the rule by which we get a mesh generating function $C^1[0, 1]$, it is as follows,

$$\varphi(t) = \begin{cases} \chi(t) = \frac{\sigma\varepsilon}{\beta} \ln \frac{q-t}{q} & for\ t \in [0, \tau], \\ \pi(t) = \chi(\tau) + \chi'(\tau)(t - \tau) & otherwise \end{cases} \quad [4]$$

where the point τ satisfies

$$\chi'(\tau) = \frac{1-\chi(\tau)}{1-\tau} \qquad (4)$$

Geometrically this implies that $(\tau, \chi(\tau))$ is the contact point of the tangent π to χ that passes through the point (1,1); see Fig. 2. When $\sigma\varepsilon \geq \rho q$, the equation (4) does not have a solution. In this case, the Bakhvalov mesh is uniform with mesh size N^{-1}. The equation (4) is nonlinear equation, its disadvantage is that, it is not solved by direct mathematical analytic methods,

$$\tau_0 = 0,\ \ \chi'(\tau_{i+1}) = \frac{1-\chi(\tau_i)}{1-\tau_i}\ , i\ =\ 0, 1, 2 \ldots$$

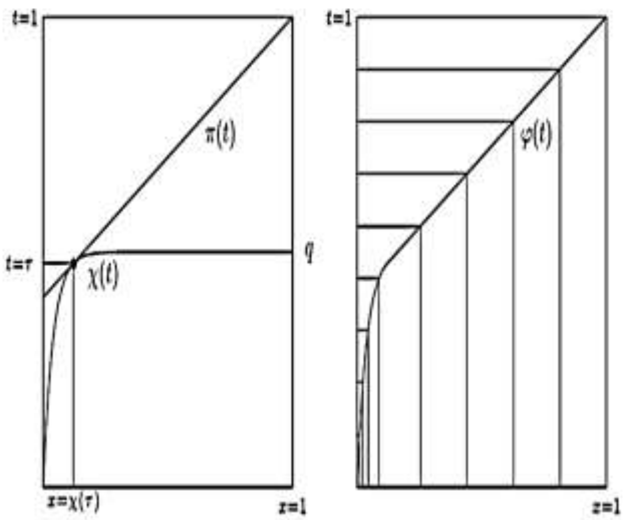

Figure 1: B-mesh: The shape on the left shows the f mesh generating function, while the other shape on the right in the above figure shows the mesh generated[4].

It should be noted that the aforementioned adjacent piecewise uniform meshes that are generated through an approximation of Bakhvalov's mesh-generating function are known as a meshes of Bakhvalov type (B-type meshes) [4].

Shishkin meshes

Shishkin meshes are piecewise uniform meshes constructed a priori that partly resolve layers. The boundary layers that appear in the solution u of equation (1) act approximately like

$$e^{-\frac{\beta_1(1-x)}{\varepsilon}} \ \ and\ \ e^{-\frac{\beta_2(1-x)}{\varepsilon}}$$

Let N be a positive integer divisible by four that denotes the number of mesh intervals used in each coordinate direction. Let τ_x and τ_y denote two mesh transition parameters defined by

$$\tau_x = \min\left(\frac{1}{2}, \tau_0 \frac{\varepsilon}{\beta_1} \ln N\right) \text{ and } \tau_y = \min\left(\frac{1}{2}, \tau_0 \frac{\varepsilon}{\beta_2} \ln N\right)$$

Whereby τ_0 is a constant that will be fixed later, ε is called the singular perturbation parameter. In pattern one typically has

$$\tau_x = \tau_0 \varepsilon {\beta_1}^{-1} \ln N \qquad (6)$$

and

$$\tau_y = \tau_0 \varepsilon {\beta_2}^{-1} \ln N \qquad (7)$$

In this paper the domain is dissected into six parts as $\bar{\Omega} = \Omega_{11} \cup \Omega_{12} \cup \Omega_{13} \cup \Omega_{21} \cup \Omega_{22} \cup \Omega_{23}$, whereby

$$\Omega_{11} = \left[0, \frac{1-\tau_x}{2}\right] \times [0, \tau_y], \Omega_{12} = \left[\frac{1-\tau_x}{2}, \frac{1+\tau_x}{2}\right] \times [0, \tau_y],$$

$$\Omega_{13} = \left[\frac{1+\tau_x}{2}, 1\right] \times [0, \tau_y],$$

$$\Omega_{21} = \left[0, \frac{1-\tau_x}{2}\right] \times [\tau_y, 1],$$

$$\Omega_{22} = \left[\frac{1-\tau_x}{2}, \frac{1+\tau_x}{2}\right] \times [\tau_y, 1], \Omega_{23} = \left[\frac{1+\tau_x}{2}, 1\right] \times [\tau_y, 1].$$

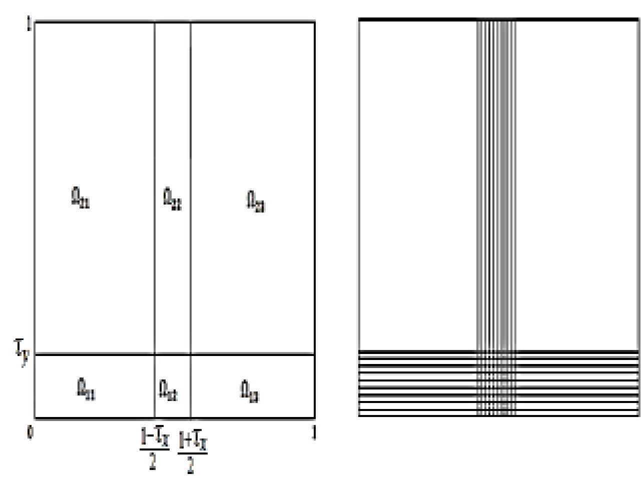

Figure 2: Dissection of **Ω** and Shishkin mesh for equation (1) , shows the regions that are formed in the mesh (left) and the fine region of the mesh (right).

A set of mesh points which are specified as follows:

$$\Omega^N = \{(x_i, y_j) \in \bar{\Omega} : i, j = 0, \ldots, N\} \qquad (5)$$

By

$$x_i = \begin{cases} \dfrac{2i(1-\tau_x)}{N} & for\ i = 0,1,\ldots,\dfrac{N}{2}, \\ 1 - \dfrac{2(N-i)\tau_x}{N} & for\ i = \dfrac{N}{2}+1,\ldots,N, \end{cases} \qquad (8)$$

And

$$y_i = \begin{cases} 1-\frac{4(N-j)\tau_y}{N} & for\ i=0,1,\dots,\frac{N}{4} \\ \frac{4j(1-\tau_y)}{N} & for\ j=\frac{N}{4},\frac{N}{4}+1,\dots,\frac{3N}{4}, \\ 1-\frac{2(N-j)\tau_y}{N} & for\ i=\frac{3N}{4},\frac{3N}{4}+1,\dots,N, \end{cases} \quad (9)$$

In Figure 1 the mesh points are detectable where the horizontal and vertical lines cross. Let Γ^Nbe the set of mesh points on the boundary of Ω i.e., $\Gamma^N = \{(x_i,y_i): i,j=0,\dots,N\}\cap\Gamma$. The mesh transition parameters τ have been chosen so that the layers have magnitude at most N^{τ_0} on the coarse mesh regions Ω_{21} and Ω_{23}. The analysis of numerical methods on Shishkin meshes shows that if a method reveals, for fixed ε, order of consistency N^{-v} on a uniform mesh. Then $\lambda_0 = v$ is a good choice. We denote by H_x, H_y, and h_x, h_y the mesh widths outside and inside the respective boundary layers, i. e.,

$H_x = 2(1-\tau_x)/N;\ h_x = \frac{2\tau_x}{N};$

$H_y = 2(1-\tau_y)/N;\ h_y = \frac{2\tau_y}{N};$

Apparently, the mesh widths on Ω_{21} and Ω_{23} satisfy $\frac{1}{N} \le H_x,\ H_y \le 2/N$, so the mesh is coarse there. On the other hand, h_x and h_y are $O(\varepsilon N^{-1}\ln N)$ so the mesh is very fine on Ω_{12}. On Ω_{11}, Ω_{22}, and Ω_{13} the mesh is coarse in one direction and convenient in the other direction [12].

5 – Definition:

If f' is continuous on [a,b], then the length (arc length) of the curve $y=f(x)$ from the point $A=(a,f(a))$ to the point $B=(b,f(b))$ is the value of the following integral [13]:

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad (10)$$

The New Mesh

The assumption of the new method we provide in this research article is inspired by the principle of the assumption of the two previous methods B-mesh in the chapter (3) and S-mesh in the chapter (4) [14], We denote the $L_\infty(\Omega)$ by,$\|.\|_\infty$ and we define an ε-infinity norm by

$$\|.\|_\infty = \max_{0\le i\le N}|x_i| \quad for\ x_i \in \Omega \qquad (11)$$

The nodes of the rectangular mesh are obtained from the concise product of a set of N points in the x direction and a set of N' points in the y direction. For notational simplicity we shall assume that $N=N'$; when this is not the case, it is easy to show that the analysis is still valid, provided only that the ratios N/N' and N'/N are bounded by some constant C. Let N be a positive integer number, divisible by four. Let s be an ideal Singular Boundary Layer (SBL) part such that $s = e^{\frac{x}{\varepsilon}}$ then by deriving with respect x we have , $s' = \frac{1}{\varepsilon}e^{\frac{x}{\varepsilon}}$, and let τ denote the length of the corresponding mesh of SBL as in the figure 3:

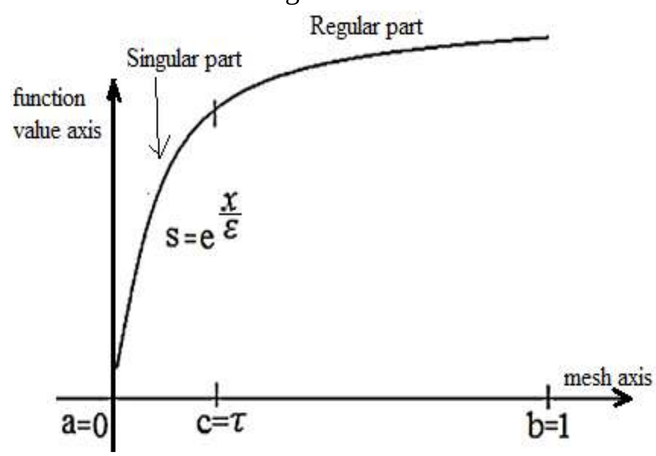

Figure 3: Explain the new mesh idea; $\boldsymbol{s}$ is a Singular Boundary Layer part such that $\boldsymbol{s} = \boldsymbol{e}^{\frac{x}{\varepsilon}}$, x-axis represents the mesh axis and the y-axis represents the range points (in two dimension plane). Also, $\boldsymbol{\tau}$ is the transition point between the fine and the coarse meshes.

We choose τ such that the length of the arc: $s\widehat{(a)f}(c)$ equal to the length of the line segment $\overline{ab}$, then assuming that $[a,b]=[0,1]$, i.e.

the length of the arc $f\widehat{(a)f}(c) = ab$

$\Longrightarrow$ the length of the arc $f\widehat{(\tau)f}(b) = 1-\tau$

$$\Rightarrow \int_0^\tau \sqrt{1+s'^2}\,dx = 1-\tau \ \{\text{by def. of arc length in chapter 5 of this paper}\}$$

$$\Rightarrow \int_0^\tau \sqrt{1+\left(\frac{1}{\varepsilon}e^{\frac{x}{\varepsilon}}\right)^2}\,dx, \qquad let\ u = \sqrt{1+\frac{e^{\frac{2x}{\varepsilon}}}{\varepsilon^2}},$$

$$du = \frac{1}{2}\left(1+\frac{e^{\frac{2x}{\varepsilon}}}{\varepsilon^2}\right)^{-\frac{1}{2}}\left(\frac{2e^{\frac{2x}{\varepsilon}}}{\varepsilon^3}\right)dx = \frac{1}{u}\frac{(u^2-1)}{\varepsilon}dx$$

$$\Rightarrow dx = \frac{\varepsilon u}{(u^2-1)}$$

When $x = 0, u = \sqrt{1+\frac{1}{\varepsilon^2}}$, and when $x = \tau$,

$$u = \sqrt{1+\frac{e^{\frac{2\tau}{\varepsilon}}}{\varepsilon^2}},$$

$$\therefore 1-\tau = \int_{\sqrt{1+\frac{1}{\varepsilon^2}}}^{\sqrt{1+\frac{e^{\frac{2\tau}{\varepsilon}}}{\varepsilon^2}}} u.\frac{\varepsilon u}{u^2-1}du = \int_{\sqrt{1+\frac{1}{\varepsilon^2}}}^{\sqrt{1+\frac{e^{\frac{2\tau}{\varepsilon}}}{\varepsilon^2}}} \frac{\varepsilon u^2}{u^2-1}du$$

$$= \int_{\sqrt{1+\frac{1}{\varepsilon^2}}}^{\sqrt{1+\frac{e^{\frac{2\tau}{\varepsilon}}}{\varepsilon^2}}} (\varepsilon - \frac{\varepsilon}{1-u^2})du$$

$$= [\varepsilon u - \varepsilon coth^{-1}(u)]_{\sqrt{1+\frac{1}{\varepsilon^2}}}^{\sqrt{1+\frac{e^{\frac{2\tau}{\varepsilon}}}{\varepsilon^2}}}$$

$$= \varepsilon\sqrt{1+\frac{e^{\frac{2\tau}{\varepsilon}}}{\varepsilon^2}} - \varepsilon coth^{-1}\left(\sqrt{1+\frac{e^{\frac{2\tau}{\varepsilon}}}{\varepsilon^2}}\right)$$

$$-\varepsilon\sqrt{1+\frac{1}{\varepsilon^2}}$$

$$+\varepsilon coth^{-1}(\sqrt{1+\frac{1}{\varepsilon^2}})$$

Finally, we get the following equation:

$$\sqrt{\varepsilon^2+e^{\frac{2\tau}{\varepsilon}}} - \varepsilon coth^{-1}\left(\sqrt{1+\frac{e^{\frac{2\tau}{\varepsilon}}}{\varepsilon^2}}\right) - \sqrt{\varepsilon^2+1} + \varepsilon coth^{-1}\left(\sqrt{1+\frac{1}{\varepsilon^2}}\right) - 1 + \tau = 0 \quad (12)$$

Then we solve the equation (12) to find the value of τ, let denote the new τ by $\tau_1 = tau$, i.e.

$$\gg \text{tau} = \text{solve}\left('\text{sqrt}\left(\varepsilon\text{\^{}}2+\exp\left(2*\frac{x}{\varepsilon}\right)\right) - \varepsilon * \text{acoth}\left(\text{sqrt}\left(1+\frac{\exp\left(2*\frac{x}{\varepsilon}\right)}{\varepsilon}\right)\right) - \text{sqrt}(\varepsilon\text{\^{}}2+1) + \varepsilon * \text{acoth}\left(\text{sqrt}(1+\varepsilon\text{\^{}}(-2))\right) - 1 + x','x'\right) \quad (13)$$

For example:

```
>>tau=solve('sqrt(0.1^2+exp(2*x/0.1))-0.1*acoth(sqrt(1+exp(2*x/0.1)/0.1))-sqrt(0.1^2+1)+0.1*acoth(sqrt(1+0.1^(-2)))-1+x','x')
```

Gives us:

tau=0.066385061798060061317993064262541

Algorithm

Step 1: Input singularly purtibation parameter ε, Number of mesh points on x-axis N_x, Number of mesh points on y-axis N_y, the PDE, the initial condition, and the boundary conditions.

Step 2: Find each of uniform mesh $x_i^U = \left\{x_i : x_i = a + \frac{i}{n}, i = 0,1,2,\dots,N_x\right\}$ and $y_i^U = \left\{y_i : y_i = a + \frac{i}{n}, i = 0,1,2,\dots,N_y\right\}$

Step 3: Find the new tau $\tau_x = min\left\{\frac{1}{2},\tau_1,\tau_2\right\}$, where τ_1 is calculated from the equation (6) by taking $\tau_0 = \beta_1 = 1$, and τ_2 is calculated from the equation (13).

New fitted piecwise mesh, and then find $\tau_y = min\left\{\frac{1}{2},\tau_3,\tau_4\right\}$, where τ_3 is calculated from the equation (7) by taking $\tau_0 = \beta_2 = 1$, and τ_4 is calculated from the equation (13).

Step 4: Construct the set of the fitted piecewise uniform mesh points $\Omega^N = \{(x_i,y_j) \in \bar{\Omega} : i,j = 0,\dots,N\}$, so that we find x_i from the equation (8) and y_j from the equation (9).

Step 5: Use the MATLAB program code (PDEPE) to solve the PDE by using once the uniform mesh input, and then the solution matrix notation is

$$z^u = \{z_{ij}^u, i = 0,1,2,\dots,N_x, j = 0,1,2,\dots,N_y\},$$

and repeat using fitted piecewise uniform mesh and the solution matrix notation is

$$z^p = \{z_{ij}^p, i = 0,1,2,\dots,N_x, j = 0,1,2,\dots,N_y\},$$

Also find the exact solution matrix let denoted by

$$z^e = \{z_{ij}^e, i = 0,1,2,\dots,N_x, j = 0,1,2,\dots,N_y\},$$

Step 6: Using the infinity norm in equation (11) to determine the errors as follows:

The error of uniform mesh ($erru$) w.r.s. the exact solution:

$$erru = \|z^u - z^e\|_\infty = \max\{\max\{|z_{ij}^u - z_{ij}^e|\}\}, i = 0,1,2,\ldots,N_x, j = 0,1,2,\ldots,N_y;$$

and the error of fitted piecewise uniform mesh ($errp$) w.r.s. the exact solution:

$$errp = \|z^p - z^e\|_\infty = \max\left\{\left|z_{ij}^p - z_{ij}^e\right|\right\}, i = 0,1,2,\ldots,N_x, j = 0,1,2,\ldots,N_y;$$

Test Problems

(**1**) Considering the following problem[15]:

$$\varepsilon\pi^2 \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2},$$

Initial condition: $u(x,0) = \sin(\pi x)$,

Boundary conditions: $u(0,y) = 0$, $\frac{\partial u}{\partial x}|_{x=1} = -\pi e^{-\frac{y}{\varepsilon}}$,

Exact Solution: $u(x,y) = \sin(\pi x)\, e^{-\frac{y}{\varepsilon}}$.

(2) Considering the following problem [16]:

$$\varepsilon \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2}, \quad \text{if } 0 \le x \le 1, 0 \le y \le 0.1,$$

Initial condition: $u(x,0) = \sin(\pi x) + \sin(2\pi x)$,

Boundary conditions: $u(0,y) = 0$, $u(1,y) = 0$,

Exact Solution: $u(x,y) = \sin(\pi x)\, e^{-\frac{\pi^2 y}{\varepsilon}} + \sin(2\pi x)\, e^{-\frac{4\pi^2 y}{\varepsilon}}$.

Results and Conclusion

The value of the ε plays a major and essential role in the solution, as its value depends inversely on the extent of the singular layers and directly with the error, as the smallness (slightness) of the ε leads to sharper in stiffness that appears in solution then leads to difficulty in the solution and a significant increase in error. In other words, the different values of the ε transform one problem into a group of distinct problems. Therefore, different values were taken, as far as possible, for the ε in each problem. Numerical comparisons between the maximum errors of the newly fitted piecewise uniform mesh-algorithm (Ft.Piec.mesh max.error) vs the maximum errors of uniform mesh (U-mesh max.error) presented, the MATLAB computer program are used. The data and results are presented through tables (1),(2),…,(7), i.e., each table contains the numerical data of two of the figures. The results of the work are presented through a set of tables; in addition, it is represented by a set of figures as follows:

Table 1: Solution of problem 1 when $\boldsymbol{\varepsilon = 0.1}$.

$\varepsilon = 0.1$		U-mesh max.error	Ft.Piec.mesh max.error	τ_x	τ_y
N_x	N_y				
72	6	0.000290983	0.00022616	0.066385062	0.387463
72	8	0.000289638	0.00022616	0.066385062	0.387463
72	10	0.000267015	0.00022616	0.066385062	0.387463
72	12	0.000290983	0.00022616	0.066385062	0.387463
72	14	0.000299768	0.00022616	0.066385062	0.387463
72	16	0.000289638	0.00022616	0.066385062	0.387463
72	18	0.000290983	0.00022616	0.066385062	0.387463
72	20	0.000299819	0.00022616	0.066385062	0.387463
72	22	0.000297625	0.000231765	0.066385062	0.387463

72	24	0.000290983	0.000239935	0.066385062	0.387463
72	26	0.00029883	0.00024593	0.066385062	0.387463
72	28	0.000299768	0.000250029	0.066385062	0.387463
72	30	0.000295963	0.000252633	0.066385062	0.387463

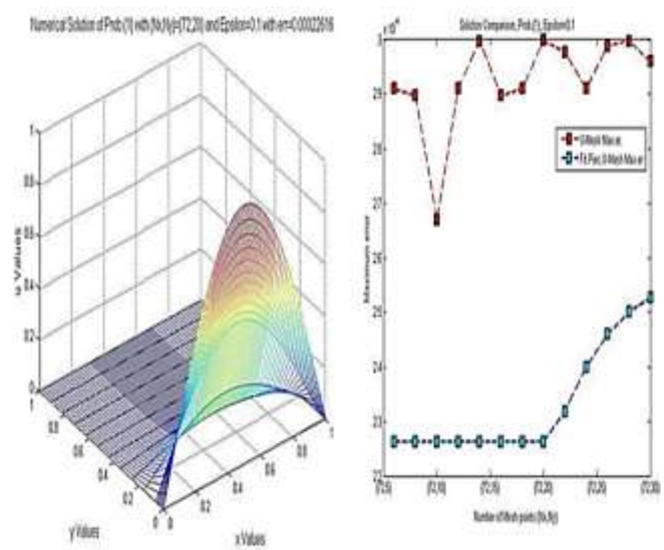

Figure 4: The solution of prob.1 when $\boldsymbol{\varepsilon = 0.1}$ (left), with the comparison of the new Fitted piecewise uniform mesh with the Uniform mesh (right).

Table 2: Solution of problem 1 when $\boldsymbol{\varepsilon = 0.05}$.

$\varepsilon = 0.05$		U-mesh max.error	Ft.Piec.mesh max.error	τ_x	τ_y
N_x	N_y				
72	34	0.000283367	0.000226965	0.387463	0.033898649
72	36	0.000290983	0.000226965	0.387463	0.033898649
72	38	0.000297056	0.000226965	0.387463	0.033898649
72	40	0.000299819	0.000226965	0.387463	0.033898649
72	42	0.000299768	0.000226965	0.387463	0.033898649
72	44	0.000297625	0.000226965	0.387463	0.033898649
72	46	0.000293831	0.000231114	0.387463	0.033898649
72	48	0.000290983	0.000235257	0.387463	0.033898649
72	50	0.000295855	0.000238943	0.387463	0.033898649
72	52	0.00029883	0.000242148	0.387463	0.033898649
72	54	0.000300042	0.000244863	0.387463	0.033898649

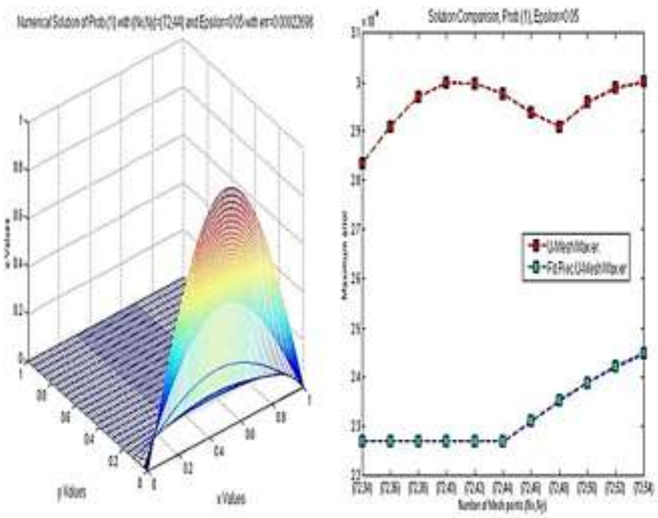

Figure 5: The solution of prob.1 when $\boldsymbol{\varepsilon} = \mathbf{0.05}$ (left), with the comparison of the new Fitted piecewise uniform mesh with the Uniform mesh (right).

Table 3: Solution of problem 1 when $\boldsymbol{\varepsilon} = \mathbf{0.01}$.

$\varepsilon = 0.01$ N_x	N_y	U-mesh max.error	Ft.Piec.mesh max.error	τ_x	τ_y
72	76	0.000294727	0.000227509	0.387463	0.006899056
72	78	0.000291965	0.000227509	0.387463	0.006899056
72	80	0.000289638	0.000227509	0.387463	0.006899056
72	82	0.000286914	0.000227509	0.387463	0.006899056
72	84	0.000284629	0.000227509	0.387463	0.006899056
72	86	0.000282055	0.000227509	0.387463	0.006899056
72	88	0.000279755	0.000227509	0.387463	0.006899056
72	90	0.000277466	0.000227509	0.387463	0.006899056
72	92	0.000275022	0.000227509	0.387463	0.006899056
72	94	0.000273121	0.000227509	0.387463	0.006899056
72	96	0.000271003	0.000227509	0.387463	0.006899056
72	98	0.00026887	0.000227509	0.387463	0.006899056
72	100	0.000267015	0.000227509	0.387463	0.006899056
72	102	0.000264943	0.000227509	0.387463	0.006899056
72	104	0.000265842	0.000227509	0.387463	0.006899056

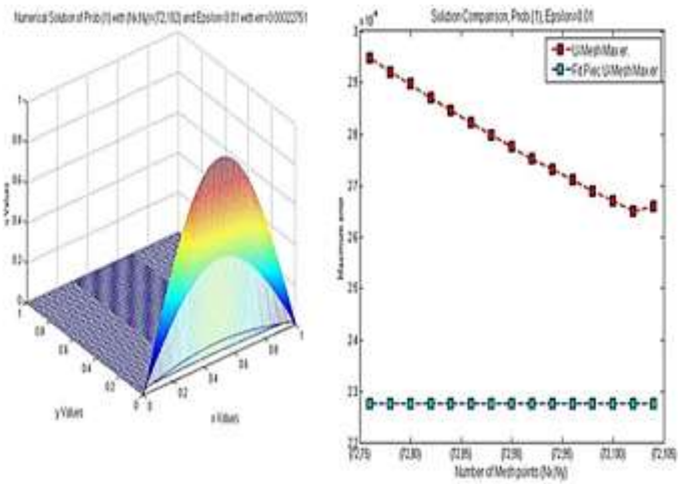

Figure 6: The solution of prob.1 when ε=0.01 (left), with the comparison of the new Fitted piecewise uniform mesh with the Uniform mesh (right).

Table 4: Solution of problem 2 when $\varepsilon = 0.5$.

$\varepsilon = 0.5$		U-mesh max.error	Ft.Piec.mesh max.error	τ_x	τ_y
N_x	N_y				
52	36	0.000634744	0.000555358	0.387463	0.05
52	38	0.000635886	0.000553717	0.387463	0.05
52	40	0.00063419	0.000553519	0.387463	0.05
52	42	0.000637299	0.000555655	0.387463	0.05
52	44	0.00063583	0.000555768	0.387463	0.05
52	46	0.000635923	0.000554833	0.387463	0.05
52	48	0.000635591	0.000553519	0.387463	0.05
52	50	0.00063724	0.000555388	0.387463	0.05
52	52	0.000636533	0.00055602	0.387463	0.05
52	54	0.000634744	0.000555358	0.387463	0.05
52	56	0.000636105	0.000554391	0.387463	0.05

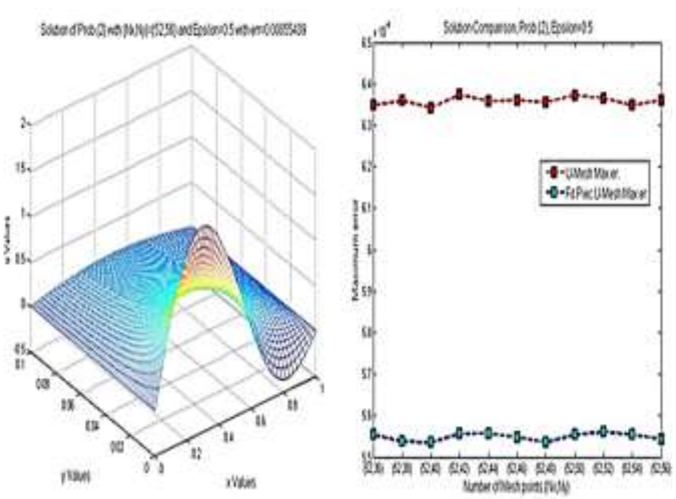

Figure 7: The solution of prob.2 when ε=0.5 (left), with the comparison of the new Fitted piecewise uniform mesh with the Uniform mesh (right).

Table 5: Solution of problem 2 when $\varepsilon = 0.1$.

$\varepsilon = 0.1$		U-mesh max.error	Ft.Piec.mesh max.error	τ_x	τ_y
N_x	N_y				
72	30	0.000278806	0.000263613	0.387463	0.034011974
72	32	0.000285787	0.00026284	0.387463	0.034657359
72	34	0.000291144	0.000261777	0.387463	0.035263605
72	36	0.000296211	0.00026089	0.387463	0.035835189
72	38	0.000299223	0.00025981	0.387463	0.036375862
72	40	0.000301708	0.000257964	0.387463	0.036888795
72	42	0.000302264	0.000255901	0.387463	0.037376696
72	44	0.000301669	0.000255822	0.387463	0.037841896
72	46	0.000300183	0.000257728	0.387463	0.038286414
72	48	0.000299211	0.000259492	0.387463	0.03871201
72	50	0.000300132	0.000260628	0.387463	0.03912023

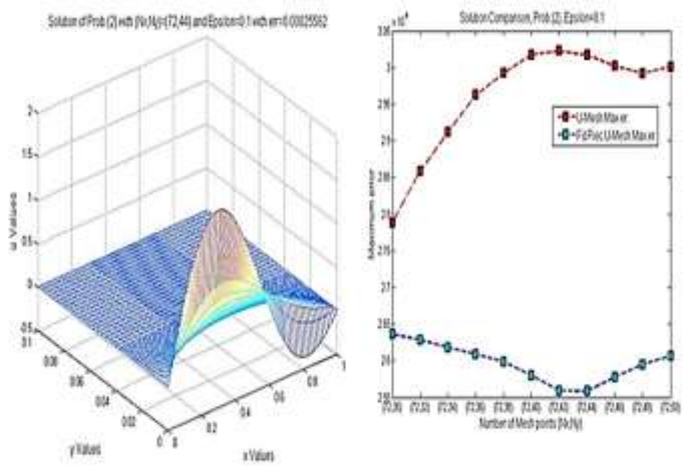

Figure 8: The solution of prob.2 when ε=0.1 (left), with the comparison of the new Fitted piecewise uniform mesh with the Uniform mesh (right).

Table 6: Solution of problem 2 when $\varepsilon = 0.05$.

$\varepsilon = 0.05$		U-mesh max.error	Ft.Piec.mesh max.error	τ_x	τ_y
N_x	N_y				
52	58	0.000592008	0.00054584	0.020302215	0.387463
52	60	0.000598869	0.000548428	0.020471723	0.387463
52	62	0.000606755	0.000550104	0.020635672	0.387463
52	64	0.000611461	0.000551329	0.020794415	0.387463
52	66	0.000617135	0.000552656	0.020948274	0.387463
52	68	0.000622286	0.000553898	0.021097539	0.387463

52	70	0.000624931	0.000554939	0.021242476	0.387463
52	72	0.000627247	0.00055572	0.021383331	0.387463
52	74	0.000631734	0.000556161	0.021520325	0.387463
52	76	0.000633451	0.000555816	0.021653667	0.387463
52	78	0.00063377	0.000555465	0.021783544	0.387463

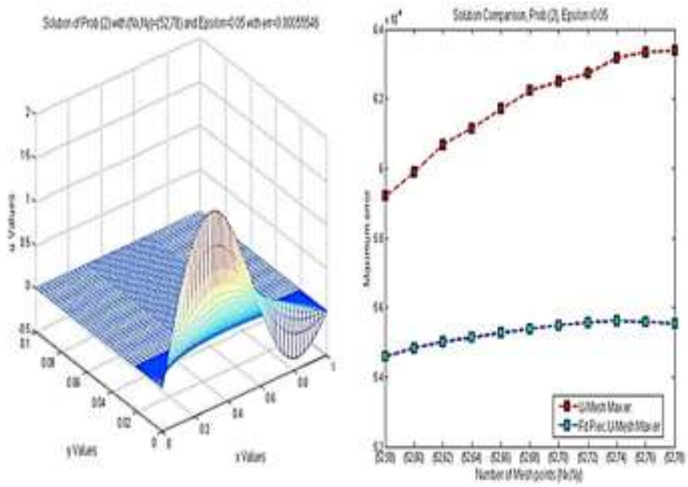

Figure 9: The solution of prob.2 when ε=0.05 (left), with the comparison of the new Fitted piecewise uniform mesh with the Uniform mesh (right).

Table 7: Solution of problem 2 when $\varepsilon = 0.01$.

$\varepsilon = 0.01$		U-mesh max.error	Ft.Piec.mesh max.error	τ_x	τ_y
N_x	N_y				
92	174	0.000138861	0.000138547	0.387463	0.000689906
92	176	0.000139838	0.00013855	0.387463	0.000689906
92	178	0.000141124	0.000138577	0.387463	0.000689906
92	180	0.000141962	0.000138542	0.387463	0.000689906
92	182	0.000142461	0.000138573	0.387463	0.000689906
92	184	0.000142709	0.000138534	0.387463	0.000689906
92	186	0.00014363	0.000138569	0.387463	0.000689906
92	188	0.000145834	0.000138525	0.387463	0.000689906
92	190	0.000146516	0.000138563	0.387463	0.000689906
92	192	0.000146073	0.000138529	0.387463	0.000689906
92	194	0.000144819	0.000138557	0.387463	0.000689906

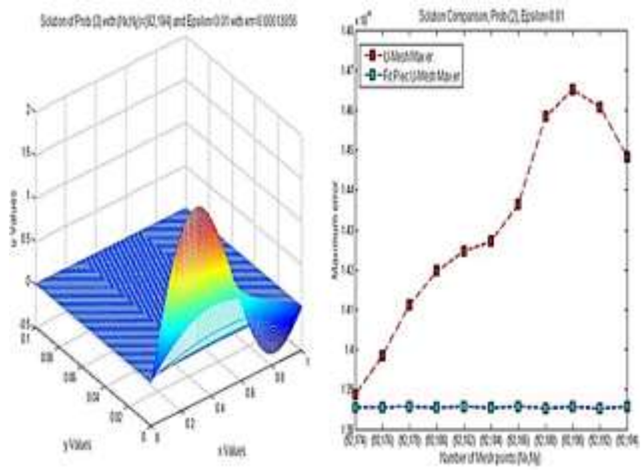

Figure 10: The solution of prob.2 when ε=0.01 (left), with the comparison of the new Fitted piecewise uniform mesh with the Uniform mesh (right).

The numerical results indicate that the new technique has an improvement about (14.82806023%) in Maximum error of the new mesh method against the uniform mesh method, as in table 8

Table 8: The percentage of improvement in Maximum error of the new-mesh method against the uniform mesh method.

ε \ Problems	Problem1	Problem2
0.5	-	12.73304385
0.1	20.31619608	12.27986919
0.05	21.36734417	10.57834542
0.01	18.22017869	3.161598101
Average percent%	19.96790631	9.68821414
Totalize percent%	14.82806023	

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مسألة ذات قيم أولية-مقيدة مع الشبكة الاعتيادية جزئيا بتقنية جديدة

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الخلاصة:

الهدف من بحثي هو إثبات الحقائق وتحديد أهميتها. تم تكوين شبكة مجزئة اعتيادية،ابسيلونتية التقارب، جديدة، من خلال اشتقاق تقنية هجينة لتوليد الفترات الجزئية (τ) للطبقات الحدودية المفردة التي تحدث عند حل بعض مسائل المعادلة التفاضلية عدديًا، حيث يتم ايجاد ε مضروبة بالحدود التي تحتوي أعلى المشتقات في المعادلة التفاضلية ، حيث يكون الغاية صفرًا ، وتكون هذه الطبقات الحدودية متاخمة لحدود المجال ، حيث ينتج عن الحل تدرج عميق جدًا . تم استخدام الشبكة باستخدام البرنامج المعروف MATLAB ، وتحديداً PDEPE الذي يحل مسائل القيم الأولية–المقيدة المتعلقة بـــ PDEs القطع المكافئ الإهليلجي. تم تطبيقه لحل أمثلة متعددة ثم مقارنة الخطأ الأكبر للحلول مع نظيره "شبكة الاعتيادية" وإثبات تفوقها. تم عرض النتائج والحلول والمقارنات بمخططات ماتلاب توضيحية موجزة تتجلى في بعض الجداول اللازمة لدراسة المقارنات.

الكلمات المفتاحية: الشبكات المجزئة الاعتيادية، المسائل المضطربة الشاذة، التقارب الابسيلونتي، المعادلات التفاضلية الجزئية