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Homotopy Analysis Method for Solving Multi-Fractional Order Random Ordinary Differential Equations

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ABSTRACT

The main objective of this paper is to present random ordinary differential equations with multi fractional derivatives and to use the homotopy analysis method to approximate the solution of such equations with different generations of the Weiner process or Brawnian motion. One of the most important and efficient methods for solving various mathematical problems with different operators, linear and nonlinear, ordinary or partial differential equations, integral equations, and so on, is the homotopy analysis method.

1. Introduction:

Ordinary differentiation and integration are generalized to an arbitrary (non-integer) order in fractional calculus. The subject is as old as differential calculus and dates back to the time when Leibnitz and Newton invented differential calculus. As a result, scientists and researchers in various fields of science and engineering have been paying close attention to fractional calculus and its applications for many years. Furthermore, due to so many nonlinear problems cannot be solved exactly, approximate and numerical methods appear to be necessary and must be used [18].

Oldham and Spainir [16], who wrote in this field or subject, began their study in 1968 with the realization that the use of half-order derivatives and integrals leads to a more economical and useful formulation of certain electrochemical problems than the classical approaches.

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This discovery stimulated our interest not only in the applications of derivative and integral notions to arbitrary order, but also in the fundamental mathematical properties of these fascinating operators.

Ordinary differential equations or partial differential equations with derivatives of any real or complex order are fractional differential equations [5]. Several authors have previously stated such equations and studied their theoretical or numerical solutions.

The He's approximation methods, which include the Homotopy Analysis Method (HAM), [1,3,8,13], Homotopy Perturbation Method (HPM), [9], Variational Iteration Method (VIM), [6,10], are among the approximate analytical methods used for solving differential equations in operator form with fractional derivatives or integrals.

Fractional random ordinary differential equations are a combination of fractional ordinary differential equations and random ordinary differential equations [14,20]. In addition to the preceding, extensive research has been devoted in recent decades to studying differential equations with random perturbations, which are sometimes referred to in stochastic calculus as a random process with specific properties [2,6,7,17].

Furthermore, random differential equations are differential equations that involve random or stochastic processes. As a result, methods dealing with such equations struggle with difficulty [12]. As consequence, random differential equations, as a subset of stochastic differential equations, are considered in this article to appear in the stochastic process without derivation.

The HAM will be used in this paper to find the approximate solution of certain types of multi-term fractional random ODEs with Caputo fractional derivatives that satisfy the existence and uniqueness theorem conditions.

2. Basic Concepts

Some preliminary information and basic concepts related to this study are provided in this section for completeness. We begin with the fundamental definitions of fractional calculus, which will be used later in the formulation of the problem in this study and its solution using the proposed approach.

Definition 1, [16]. Let $y: [a,b] \longrightarrow \mathbb{R}$ be a function, α a positive real number, n a positive integer satisfying $n-1<\alpha\leq n$ and Γ is the gamma function. The left and right Riemann-Liouville fractional integrals of order α are given respectively by:

$$ {}^{RL}_{a}I^{\alpha}_{t}\, y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} y(s)\, ds$$

$$ {}^{RL}_{b}I^{\alpha}_{t}\, y(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1} y(s)\, ds,$$

where $y\in C^n[a,b]$, $\alpha\geq -1$, $\beta\geq 0$, $\alpha+\beta\leq n$ and $t\in[a,b]$.

In fractional differential equations and because of the occurrence of the initial conditions, the left Riemann-Liouville fractional integral will be used, which is therefore will abbreviated as ${}^{RL}I^{\alpha}_{t}$ in this study.

Definition 2, [4,22]. The Caputo fractional order derivative of a suitable function $y\in C^n[a,b]$ is:

$$ {}^{C}D^{\alpha}_{t} y(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} y^{(n)}(s)\, ds,$$

for $t\in[a,b]$, $\alpha\in\mathbb{R}^+$ and $n-1<\alpha\leq n$, $n\in\mathbb{N}$.

Some of the most important properties of fractional order derivative and integrals may be summarized in the next [2,6,10,16]:

1. If $n-1<\alpha\leq n$, $n\in\mathbb{N}$ and y is any function then ${}^{RL}I^{\alpha}_{t}\,{}^{C}D^{\alpha}_{t} y(t) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{t^k}{k!}$.
2. ${}^{C}D^{\alpha}_{t}\,{}^{RL}I^{\alpha}_{t} y(t) = y(t)$.
3. ${}^{C}D^{\alpha}_{t} t^{v} = \frac{\Gamma(v+1)}{\Gamma(V-\alpha+1)} t^{v-\alpha}$, for $v>-1$, $\alpha\in\mathbb{R}^+$.
4. ${}^{RL}I^{\alpha}_{t} t^{v} = \frac{\Gamma(v+1)}{\Gamma(V+\alpha+1)} t^{\alpha+v}$, for $v>0$, $\alpha\in\mathbb{R}^+$.
5. ${}^{RL}I^{0}_{t} y(t) = {}^{C}D^{0}_{t} y(t) = y(t)$.
6. If $y^{(i)}(0)=0$, $i=0,1,\cdots,n-1$, $n\in\mathbb{N}$ and if $\alpha+\beta\leq n$, $\alpha,\beta\in\mathbb{R}^+$ then:
   i. ${}^{C}D^{\alpha}_{t}\,{}^{C}D^{\beta}_{t} y(t) = {}^{C}D^{\beta}_{t}\,{}^{C}D^{\alpha}_{t} y(t) = {}^{C}D^{\alpha+\beta}_{t} y(t)$.
   ii. ${}^{C}D^{\alpha}_{t}\,{}^{RL}I^{\beta}_{t} y(t) = {}^{RL}I^{\beta}_{t}\,{}^{C}D^{\alpha}_{t} y(t) = {}^{C}D^{\alpha-\beta}_{t} y(t) = {}^{RL}I^{\beta-\alpha}_{t} y(t)$.
7. ${}^{C}D^{\alpha}_{t}(a_1y_1+a_2y_2) = a_1\,{}^{C}D^{\alpha}_{t}(y_1) + a_2\,{}^{C}D^{\alpha}_{t}(y_2)$, $\alpha\in\mathbb{R}^+$, $a_1,a_2\in\mathbb{R}$.
8. ${}^{RL}I^{\alpha}_{t}(a_1y_1+a_2y_2) = a_1\,{}^{RL}I^{\alpha}_{t}(y_1) + a_2\,{}^{RL}I^{\alpha}_{t}(y_2)$, $\alpha\in\mathbb{R}^+$, $a_1,a_2\in\mathbb{R}$.

Stochastic calculus, which is related to this study, is a branch of mathematics that deals with random (or chance) occurrences in which an experiment occurs with finite or infinite possible outcomes. As a result, it is necessary to first explain the meaning of the following notations: sample space is the collection of all possible results of a random experiment, and it is represented by Ω.

In set language, the sample space is known as the universal set; thus, the sample space Ω is a set consisting of a mutually exclusive, collectively exhaustive collection of all potential results of a random experiment. That is, $\Omega=\{\omega_1,\omega_2,\cdots,\omega_n\}$ denotes the set of all finite outcomes, whereas $\Omega=\{\omega_1,\omega_2,\cdots\}$ denotes the set of all countably infinite outcomes, and denotes the set of unaccountably outcomes. A random variable is also a real-valued function $x(\omega)$, $\omega\in\Omega$ that can be measured with the probability measure P [2,6,19].

When stochastic processes occur, stochastic and random differential equations appear, which have many types, one of which is called the Wiener process or the Brownian motion, which is used in this paper and can be defined as follows:

Definition 3, [12]. Let (Ω, F, P) be a probability space. A stochastic process $\omega_t, t \in [0, \infty)$, is said to be a Brownian motion or Wiener process, if:

1. $P(\{\omega_0 \in \Omega \mid \omega_0 = 0\}) = 1$.
2. For $0 < t_0 < t_1 < \cdots < t_N$, the increments $\omega_{t_1} - \omega_{t_0}$, $\omega_{t_2} - \omega_{t_1}$, …, $\omega_{t_N} - \omega_{t_{N-1}}$ are independent, for any $N \in \mathbb{N}$.
3. For an arbitrary t and $h > 0$, $\omega_{t+h} - \omega_t$ has a Gaussian distribution with mean 0 and variance h.

where F stands for the σ-algebra of subsets of a sample space Ω and P for a probability measure.

Among the main objectives of this article is to find the solution of the following multi-fractional order random ODE:

$$ {}^{C}D_t^{\alpha} y_t(\omega_t) = f(t, \omega_t, y_t(\omega_t), {}^{C}D_t^{\beta} y_t(\omega_t)), \quad \ldots(1)$$

for all $t \in [0, T]$, $T \in \mathbb{R}^+$ and with initial conditions:

$$y_0^{(i)}(\omega_0) = y_0^i, i = 0,1, \cdots, n-1,$$

where ${}^{C}D_x^{\alpha}$, ${}^{C}D_x^{\beta}$ are the Caputo fractional order derivatives of order α, such that $n - 1 < \alpha \le n$, $0 < \beta < \alpha$, $n \in \mathbb{N}$, $y_0^{(i)}$ are given initial conditions and f is any given continuously differentiable function with respect to y_t.

3. Existence and Uniqueness of Solution of Fractional Order Random Ordinary Differential Equations

In this section, we will state and prove the existence and uniqueness theorem of eq. (1) using Schauder fixed point theorem. For the purpose of simplicity, the proof will be carried out for $\alpha \in (0,1]$.

Theorem 1. Let $f: [0, T] \times \Omega^2 \longrightarrow \mathbb{R}$ be a function which satisfies:

i. $f(t, \omega_t, y_t(\omega_t), {}^{C}D_t^{\beta} y_t(\omega_t))$ is Lebesgue measurable with respect to $t \in [0, T]$.

ii. $f(t, \omega_t, y_t(\omega_t), {}^{C}D_t^{\beta} y_t(\omega_t))$ is continuous with respect to $t \in [0, T]$.

iii. There exists a constant $c \in (0, \alpha)$ and a real valued function $m(t)$ which belongs to the Banach space $L^{\frac{1}{c}}([0, T])$, $\frac{1}{c} \ge 1$ of all continuous functions on $[0, T]$ with the norm defined by $\|m\|_{1/c} = \left(\int_0^T |m(s)|^{\frac{1}{c}} ds\right)^c$ such that $\left\|f(t, \omega_t, y_t(\omega_t), {}^{C}D_t^{\beta} y_t(\omega_t))\right\| \le m(t)$, for all $t \in [0, T]$.

Then for any $\alpha \in (0,1]$, there exist at least one solution of the fractional order random ODE (1) on $[-h, h]$, where:

$$h = \min\left\{a, \frac{c\Gamma(\alpha)}{M}\left(\frac{\alpha - c}{1 - c}\right)^{1-c}\right\}^{\frac{1}{\alpha - c}}$$

$$M = \left(\int_0^a (m(s))^{\frac{1}{c}} ds\right)^c.$$

Proof. From i, whenever f is Lebesgue measurable, then the integral equation equivalently related to the fractional order random ODE (1) is:

$$y_t(\omega_t) = y_0(\omega_0) + \frac{1}{\Gamma(\alpha)} \int_0^t f(s, \omega_s, y_s(\omega_s), {}^{C}D_s^{\beta} y_s(\omega_s))\, ds.$$

Since $\alpha \in (0,1]$, i.e., the function is at most has first order derivative, then for $y_t \in C^1([0, T], \mathbb{R})$ and define the norm over the Banach space $C^1([0, T], \mathbb{R})$ to be the supremum norm over the region $D = \{y_t \in C^1([0, T], \mathbb{R}): \|y_t - y_0\| \le b\}$, which is closed and bunded. Define the following operator:

$$T(y_t(\omega_t)) = y_0(\omega_0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, \omega_s, y_s(\omega_s), {}^{C}D_s^{\beta} y_s(\omega_s))\, ds.$$

By using Hölder inequality it is obtained that $(t - s)^{\alpha - 1} f(s, \omega_s, y_s(\omega_s), {}^{C}D_s^{\beta} y_s(\omega_s))$ is Lebsegue integrable with respect to $s \in [0, t]$, for all $t \in [0, h]$ and

$$\int_0^t \left\|(t - s)^{\alpha - 1} f(s, \omega_s, y_s(\omega_s), {}^{C}D_s^{\beta} y_s(\omega_s))\right\| ds \le \left(\int_0^t ((t - s)^{\alpha - 1})^{\frac{1}{1-c}} ds\right)^{1-c} \left(\int_0^t (m(s))^{\frac{1}{c}} ds\right)^c. \quad \ldots(2)$$

Now, to show that $T(y_t) \in D$, for any $y_t \in D$

By Hölder inequality and condition iii, we obtain that

$$\|T(y_t)-y_0\| = \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))ds\right\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))\right\| ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s)\,ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t ((t-s)^{\alpha-1})^{\frac{1}{1-c}}ds\right)^{1-c}\left(\int_0^t (m(s))^{\frac{1}{c}}ds\right)^{c},$$

from inequality (2)

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-c}}ds\right)^{1-c}\left(\int_0^t (m(s))^{\frac{1}{c}}ds\right)^{c}$$

$$= \frac{1}{\Gamma(\alpha)}\left(\left.\frac{-(t-s)^{\frac{\alpha-1}{1-c}+1}}{\frac{\alpha-1}{1-c}+1}\right|_0^t\right)^{1-c}\left(\int_0^t (m(s))^{\frac{1}{c}}ds\right)^{c}$$

$$= \frac{1}{\Gamma(\alpha)}\left(\left.\frac{-(t-s)^{\frac{\alpha-c}{1-c}}}{\frac{\alpha-c}{1-c}}\right|_0^t\right)^{1-c}\left(\int_0^t (m(s))^{\frac{1}{c}}ds\right)^{c}$$

$$= \frac{1}{\Gamma(\alpha)}\left(\frac{1-c}{\alpha-c}\right)^{1-c} t^{\alpha-c}\int_0^t \left(m(s)^{\frac{1}{c}}ds\right)^{c}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\frac{1-c}{\alpha-c}\right)^{1-c} T^{\alpha-c}M \leq 1, \text{ for all } t\in[0,T],$$

so $\|T(y_t)-y_0\| \leq b$ and therefore $T(y_t)\in D$.

Now, we have to show T is continuous for any $y_{m_t}, y_t \in D$, $m = 1,2,\cdots$ and since $\lim_{m\to 0}\|y_{m_t}-y_t\| = 0$, then $\lim_{m\to 0} y_{m_t}(\omega_t) = y_t(\omega_t)$, for $t\in[0,T]$.

Thus, by condition ii, we have:

$$\lim_{m\to\infty} f(t,\omega_t,y_{m_t}(\omega_t),\ {}^{C}D_t^{\beta} y_{m_t}(\omega_t)) = f(t,\omega_t,y_t(\omega_t),\ {}^{C}D_t^{\beta} y_t(\omega_t)),$$

and hence as $m\longrightarrow\infty$

$$\sup_{t\in[0,T]}\left\|f(t,\omega_t,y_{m_t}(\omega_t),\ {}^{C}D_t^{\beta} y_{m_t}(\omega_t)) - f(t,\omega_t,y_t(\omega_t),\ {}^{C}D_t^{\beta} y_t(\omega_t))\right\| \longrightarrow 0. \quad \ldots(3)$$

So:

$$\|T(y_{m_t})-T(y_t)\| = \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}\left(f(s,\omega_s,y_{m_s}(\omega_s),\ {}^{C}D_s^{\beta} y_{m_s}(\omega_s)) - f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))\right)\right\| ds$$

$$\leq \frac{h^{\alpha}}{\Gamma(\alpha+1)}\sup_{t\in[0,T]}\left\|f(t,\omega_t,y_{m_t}(\omega_t),\ {}^{C}D_t^{\beta} y_{m_t}(\omega_t)) - f(t,\omega_t,y_t(\omega_t),\ {}^{C}D_t^{\beta} y_t(\omega_t))\right\|.$$

Hence, from (3), getting $\|T(y_{m_t})-T(y_t)\| \longrightarrow 0$ as $m\longrightarrow\infty$, i.e., T is a continuous operator.

Now, to show that T is compact, i.e., to show that the family of functions $\{T(y_t): y_t\in D\}$ is uniformly bounded and equicontinuous on D, i.e., to show that T is compact for all $y_t\in D$. We get $\|T(y_t)\| \leq \|y_t\| + b$, i.e., $\{T(y_t): y_t\in D\}$ is uniformly bounded and for any $t_1,t_2\in[0,T]$, $t_1<t_2$, by using the Hölder inequality, we have:

$$\|T(y_{t_2})-T(y_{t_1})\| = \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_2}(t_2-s)^{\alpha-1} f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))ds - \int_0^{t_1}(t_1-s)^{\alpha-1} f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))ds\right\|$$

$$= \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}(t_2-s)^{\alpha-1} f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1} f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))ds - \int_0^{t_1}(t_1-s)^{\alpha-1} f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))ds\right\|$$

$$= \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}] f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1} f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))ds\right\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left\|[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}] f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))\right\| ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left\|(t_2-s)^{\alpha-1} f(s,\omega_s,y_s(\omega_s),\ {}^{C}D_s^{\beta} y_s(\omega_s))\right\| ds$$

$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-\alpha)^{\alpha-1}]m(s)\,ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}m(s)ds$

$\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]\,ds\right)^{\frac{1}{1-c}}\left(\int_0^{t_1}(m(s))^{\frac{1}{c}}ds\right)^c+\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right)^{\frac{1}{1-c}}\left(\int_{t_1}^{t_2}(m(s))^{\frac{1}{c}}\right)^c$

$\leq \frac{2M}{\Gamma(\alpha)}\left(\frac{1-c}{\alpha-1}\right)^{1-c}(t_2-t_1)^{\alpha-c}.$

As $t_1 \longrightarrow t_2$, then $\{T(y_t): y_t \in D\}$ is eqicontinuous on $[0,T]$ and hence T is compact.

By Schauder fixed point theorem, there exists $y_t^* \in D$, such that $Ty_t^* = y_t^*$, which means that y_t^* is a fixed point of the operator T and hence y_t^* is a solution of fractional random ODE (1) over $[0,T]$. ■

It is worth noting that when f satisfies the Lipschtiz condition, the solution is unique, and the proof of Theorem 1 can be proved for and $\alpha \in (n-1,n]$, but the proof is more advanced.

4. Application of the HAM for Multi-Fractional Order Random ODEs

Several authors have successfully used the HAM as an operator equation to solve a wide range of nonlinear problems in science and engineering [6,11,13,15,21]. To begin using this method to solve multi-fractional order random ODEs, consider the general form of this equation in operators form:

$$N[y_t(\omega_t)] = 0, \quad \ldots(4)$$

where N is a nonlinear operator and y_t is the unknown function to be determined as the solution of problem (4).

Suppose that $y_{0_t}(\omega_t)$ is the initial guess approximate solution of the exact solution of eq. (4), $h \neq 0$ be an auxiliary parameter, $H(t) \neq 0$ an auxiliary function and L an auxiliary linear operator with property:

$$L[y_t(\omega_t)] = 0, \text{ when } y_t(\omega_t) = 0. \quad \ldots(5)$$

Construct using $p \in [0,1]$ as an embedding parameter, the so called zero-order deformation:

$$L[Q_t(\omega_t,p) - y_{0_t}(\omega_t)] = phH(t)N[Q_t(\omega_t,p)], \quad \ldots(6)$$

where Q_t is the solution of the operator equation which depends on h, $H(t)$, L, $y_{0_t}(\omega_t)$ and p, where $p = 0$, the zero-order deformation given by eq. (6) becomes $L[Q_t(\omega_t,p) - y_{0_t}(\omega_t)] = 0$, and so $L[Q_t(\omega_t,p)] = L[y_{0_t}(\omega_t)]$. Then taking L^{-1}, will implies to:

$$Q_t(\omega_t,p) = y_{0_t}(\omega_t), \quad \ldots(7)$$

and when $p = 1$, since $h \neq 0$ and $H(t) \neq 0$ the zero-order deformation (6) becomes:

$$N[Q_t(\omega_t,1)] = 0. \quad \ldots(8)$$

As a result, $Q_t(\omega_t,p)$ is the solution of the nonlinear equation (4) that defines the m^{th}-order deformation derivatives:

$$y_{m_t}(\omega_t) = \frac{1}{m!}\frac{\partial^m Q_t(\omega_t,p)}{\partial p^m}\bigg|_{p=0}, \ m = 1,2,\cdots \quad \ldots(9)$$

If $y_{m_t}(\omega_t)$ in eq. (9) exist at $p = 1$ for all values of m, then we get the following series solution when expanding eq. (10) using Taylor series expansion:

$$y_t(\omega_t) = Q_t(\omega_t,1) = y_{0_t}(\omega_t) + \sum_{m=1}^{\infty} y_{m_t}(\omega_t). \quad \ldots(10)$$

The previously mentioned equation grants us with correlation between the exact solution $y_t(\omega_t)$ and the initial guess approximation $y_{0_t}(\omega_t)$ with aid of the expression $y_{m_t}(\omega_t)$, $m = 1,2,\cdots$, which are unknown till the present stage.

The higher-order deformation equation of the next iterated solutions may be derived by first defining the vector:

$$\vec{y}_{i_t}(\omega_t) = [y_{0_t}(\omega_t), y_{1_t}(\omega_t), \cdots, y_{i_t}(\omega_t)] \quad \ldots(11)$$

Differentiating eq. (6) m-times with respect to the embedding parameter p and dividing by $m!$ after setting $p = 0$, we have the so-called m^{th}-order deformation equation:

$$L[y_{m_t}(\omega_t,p) - \chi_m y_{m-1_t}(\omega_t,p) = hH(t)R_m(\vec{y}_{m-1_t}(\omega_t)), \quad \ldots(12)$$

where:

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad \ldots(13)$$

$$R_m(\vec{y}_{m-1_t}(\omega_t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[Q_t(\omega_t,p)]}{\partial p^{m-1}}\bigg|_{p=0} \quad \ldots(14)$$

Thus, we can get $y_{0_t}(\omega)$, $y_{1_t}(\omega)$, … to be the high order deformation equation (12) one after one in ascending order. Finally, the m^{th}-order approximate solution of eq. (14) is given by:

$$y_t(\omega_t) = \sum_{i=0}^{\infty} y_{i_t}(\omega_t). \quad \ldots(15)$$

The proposed study of this section is to apply the HAM to solve random multi fractional order ODE, which is presented in eqs. (1) which is proceeded by considering:

$$N[y_t(\omega_t,p)] = {}^{C}D_t^{\alpha} y_t(\omega_t,p) - f(t,\omega_t,y_t(\omega_t), {}^{C}D_t^{\beta} y_t(\omega_t)), \quad \ldots(16)$$

and hence the approximated unknown function y_t can be evaluated as in the above approach.

5. Convergence Analysis

To prove the convergence of the approximate solution of the multi-term random fractional order ODE (1) presented in Section 4 to the exact solution. It is interesting that, as long as the series (15) converges, it can be concluded that:

$$\sum_{m=1}^{\infty} R_m(\vec{y}_{m-1_t}(\omega_t)) = 0.$$

If the series $\sum_{m=0}^{\infty} y_{m_t}(\omega_t,p)$ is convergent, then it can be described as:

$$S_t(\omega_t) = \sum_{m=0}^{k} y_{m_t}(\omega_t,p),$$

and it holds that:

$$\lim_{m\to\infty} y_{m_t}(\omega_t,p) = 0, \quad \ldots(17)$$

using eq. (13) and the left-hand side of eq. (12), then:

$$\sum_{m=1}^{k}[y_{m_t}(\omega_t,p) - \chi_m y_{m-1_t}(\omega_t)] = y_{1_t}(\omega_t,p) + y_{2_t}(\omega_t,p) - y_{1_t}(\omega_t,p) + y_{3_t}(\omega_t,p) - y_{2_t}(\omega_t,p) + \cdots + y_{k_t}(\omega_t,p) - y_{k-1_t}(\omega_t,p) = y_{k_t}(\omega_t,p),$$

so according to eq. (17), we have:

$$\sum_{m=1}^{\infty}[y_{m_t}(\omega_t,p) - \chi_m y_{m-1_t}(\omega_t)] = \lim_{m\to\infty} y_{m_t}(\omega_t,p) = 0.$$

Hence, using the linear operator $L = {}^{C}D_t^{\alpha}$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and the related properties of fractional calculus, one may get:

$$\sum_{m=1}^{\infty} L[y_{m_t}(\omega_t,p) - \chi_m y_{m-1_t}(\omega_t)] = L\sum_{m=1}^{\infty}[y_{m_t}(\omega_t,p) - \chi_m y_{m-1_t}(\omega_t)] = 0,$$

and eq. (12) satisfies:

$$\sum_{m=1}^{\infty} L[y_{m_t}(\omega_t,p) - \chi_m y_{m-1_t}(\omega_t)] = H_t(\omega_t) h \sum_{m=1}^{\infty} R_m(\vec{y}_{m-1_t}(\omega_t)) = 0,$$

and since $h \neq 0$, $H_t(\omega_t) \neq 0$, then:

$$\sum_{m=1}^{\infty} R_m(\vec{y}_{m-1_t}(\omega)) = 0.$$

Substituting eqs. (16) and (14) into the previous equation and by reducing it, since the series (15) is convergent, then we have:

$$\sum_{m=1}^{\infty} R_m(\vec{y}_{m-1_t}(\omega_t)) = \sum_{m=1}^{\infty}\left[{}^{C}D_t^{\alpha} y_{m-1_t}(\omega_t) + {}^{C}D_t^{\beta} y_{m-1_t}(\omega_t) + y_{m-1_t}(\omega_t) - (1-\chi_m) g_t(\omega_t)\right]$$
$$= \sum_{m=1}^{\infty} {}^{C}D_t^{\alpha} y_{m-1_t}(\omega_t) + \sum_{m=1}^{\infty} {}^{C}D_t^{\beta} y_{m-1_t}(\omega_t) + \sum_{m=1}^{\infty} y_{m-1_t}(\omega_t) - \sum_{m=1}^{\infty}(1-\chi_m) g_t(\omega_t)$$
$$= {}^{C}D_t^{\alpha}\sum_{m=0}^{\infty} y_{m_t}(\omega_t) + {}^{C}D_t^{\beta}\sum_{m=0}^{\infty} y_{m_t}(\omega_t) + S_t(\omega_t) - g_t(\omega_t) = 0$$

where is the nonhomogeneous term and form initial conditions and eq. (10), getting:

$$S_t(\omega_t) = \sum_{m=0}^{\infty} y_{m_t}(\omega_t).$$

Thus $S_t(\omega_t)$ satisfy eq. (1) and it must be the exact solution for the initial value problem (1).

6. Numerical Simulation

In this section, three examples will be considered and solved by simulating 1000 and 10000 generations of Brownian motions.

Example 1. Consider the linear multi-fractional order random ODE:

$${}^{C}D_t^{2.7} y_t(\omega_t) + {}^{C}D_t^{0.4} y_t(\omega_t) = \sin(\omega_t), t \in [0,1], \ldots(18)$$

subject to the initial condition $y_0(\omega_0) = 1$.

To start the solution, consider a fixed Brownian motion and let:

$$g_t(\omega_t) = \sin(\omega_t), \quad \ldots(19)$$

and consider the initial guess solution $y_{0_t}(\omega_t) = y_0(\omega_0) = 1$. Hence:

$$N[y_t(\omega_t)] = {}^{C}D_t^{2.7}y_t(\omega_t) + {}^{C}D_t^{0.4}y_t(\omega_t) - \sin(\omega_t). \quad \ldots(20)$$

Let $L = {}^{C}D_t^{2.7}y_t(\omega_t)$, $h = -1$ and $H_t(\omega_t) = 1$. Thus, according to eq. (18),

$${}^{C}D_t^{2.7}[y_{m_t}(\omega_t) - \chi_m y_{m-1_t}(\omega_t)] = -R_m(\vec{y}_{m-1_t}(\omega_t)) \quad \ldots(21)$$

$$R_m(\vec{y}_{m-1_t}(\omega_t)) = {}^{C}D_t^{2.7}y_{m-1_t}(\omega_t) + {}^{C}D_t^{0.4}y_{m-1_t}(\omega_t) - (1 - \chi_m)\sin(\omega_t). \quad \ldots(22)$$

The zero-order deformation is:

$${}^{R}I_t^{2.7}\,{}^{C}D_t^{2.7}[y_{1_t}(\omega_t) - \chi_1 y_{0_t}(\omega_t)] = -{}^{R}I_t^{2.7}R_1(y_{0_t}(\omega_t)),$$

where:

$$R_1(y_{0_t}(\omega_t)) = {}^{C}D_t^{2.7}y_{0_t}(\omega_t) + {}^{C}D_t^{0.4}y_{0_t}(\omega_t) + (1 - \chi_1)\sin(\omega_t)$$
$$= {}^{C}D_t^{2.7}y_{0_t}(\omega_t) + {}^{C}D_t^{0.4}y_{0_t}(\omega_t) + \sin(\omega_t).$$

Now, applying Riemann-Liouville fractional integral of order 2.7 to the both sides of eq. (21) and using the initial approximate solution (19), then the functions $y_{1_t}(\omega_t)$ may be evaluated as:

$$y_{1_t}(\omega_t) = -{}^{R}I_t^{2.7}R_1(y_{0_t}(\omega_t))$$
$$= -{}^{R}I_t^{2.7}[{}^{C}D_t^{2.7}y_{0_t}(\omega_t) + {}^{C}D_t^{0.4}y_{0_t}(\omega_t) + \sin(\omega_t)]$$
$$= -[{}^{R}I_t^{2.7}\,{}^{C}D_t^{2.7}y_{0_t}(\omega_t) + {}^{R}I_t^{2.7}\,{}^{C}D_t^{0.4}y_{0_t}(\omega_t) + {}^{R}I_t^{2.7}(\sin(\omega_t))]$$
$$= -[1 + {}^{R}I_t^{2.3}(1) + {}^{R}I_t^{2.7}(\sin(\omega_t))]$$
$$= -1 - \frac{1}{\Gamma(3.3)}t^{2.3} - \frac{\sin(\omega_t)}{\Gamma(3.7)}t^{2.7}.$$

The higher order deformation is started by letting $m = 2$, then:

$${}^{R}I_t^{2.7}\,{}^{C}D_t^{2.7}[y_{2_t}(\omega_t) - \chi_2 y_{1_t}(\omega_t)] = -{}^{R}I_t^{2.7}R_2(y_{1_t}(\omega_t)),$$

where:

$$R_2(y_{1_t}(\omega_t)) = {}^{C}D_t^{2.7}y_{1_t}(\omega_t) + {}^{C}D_t^{0.4}y_{1_t}(\omega_t) - (1 - \chi_2)\sin(\omega_t)$$
$$= {}^{C}D_t^{2.7}y_{1_t}(\omega_t) + {}^{C}D_t^{0.4}y_{1_t}(\omega_t).$$

Hence:

$$y_{2_t}(\omega_t) = y_{1_t}(\omega_t) - {}^{R}I_t^{2.7}(R_2(y_{1_t}(\omega_t)))$$
$$= y_{1_t}(\omega_t) - {}^{R}I_t^{2.7}({}^{C}D_t^{2.7}y_{1_t}(\omega_t) + {}^{C}D_t^{0.4}y_{1_t}(\omega_t))$$
$$= y_{1_t}(\omega_t) - y_{1_t}(\omega_t) - {}^{R}I_t^{2.3}(y_{1_t}(\omega_t))$$
$$= -{}^{R}I_t^{2.3}\left(1 - \frac{1}{\Gamma(3.3)}t^{2.3} - \frac{\sin(\omega_t)}{\Gamma(3.7)}t^{2.7}\right)$$
$$= \frac{1}{\Gamma(3.3)}t^{2.3} + \frac{1}{\Gamma(5.6)}t^{4.6} + \frac{\sin(\omega_t)}{\Gamma(6)}t^{5}.$$

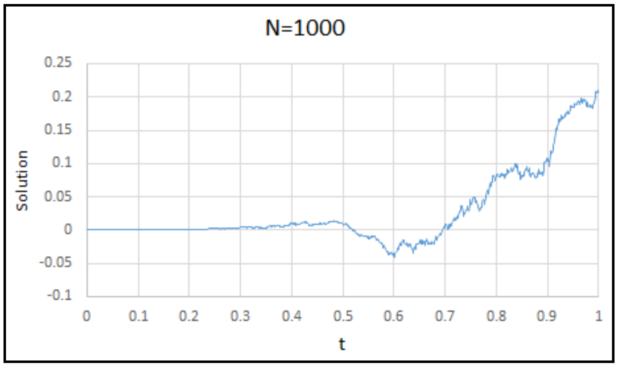
Also, if $m = 3$, then applying similarly as in the above, getting:

$$y_{3_t}(\omega_t) = -0.000107742t^{7.3}\sin(\omega_t) - 0.000242559t^{6.9} - 0.0162459t^{4.6}.$$

So on, one may proceed to find $y_{4_t}(\omega_t)$, $y_{5_t}(\omega_t)$, … and thus the solution is given by:

$$y_t(\omega_t) = y_{0_t}(\omega_t) + y_{1_t}(\omega_t) + y_{2_t}(\omega_t) + \cdots$$
$$= 1.55851 \times 10^{-8}t^{4.6} - 0.000242559t^{6.9} + (-0.2397712.7 + 0.00833333t^{5} - 0.000107742t^{7.3})\sin(\omega_t). \quad \ldots(23)$$

Figures 1 presents the approximate solution (23) of eq. (18) in terms of $y_{0_t}(\omega_t)$, $y_{1_t}(\omega_t)$, $y_{2_t}(\omega_t)$ and $y_{3_t}(\omega_t)$ with 1000 and 10000 generations of Brownian motion.



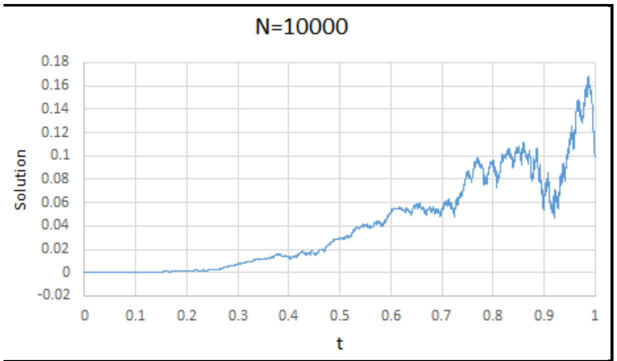


Figure 1. The approximate solution of Example 1 using the HAM for different number of Brownian motions 1000 and 10000, respectively.

Example 2. Consider the linear multi-term fractional order random ODE:

$$ {}^{C}D_t^{1.4} y_t(\omega_t) + \omega^2\, {}^{C}D_t^{0.3} y_t(\omega_t) = g_t(\omega_t), \quad \ldots(24)$$

for all $t \in [0,1]$ subject to the initial condition $y_0(\omega_0) = y_0'(\omega_0) = 0$, and $g_t(\omega_t) = \frac{\Gamma(3)}{\Gamma(1.6)} t^{0.6} + \frac{\Gamma(3)}{\Gamma(2.7)} \omega_t^2 t^{1.7}$.

First, choosing the first guest approximation $y_{0_t}(\omega_t) = 0$, and hence

$N_t[y_t(\omega_t)] = {}^{C}D_t^{1.4} y_t(\omega_t) + \omega^2\, {}^{C}D_t^{0.3} y_t(\omega_t) - g_t(\omega_t)$,

then:

$$ L\left[y_{m_t}(\omega_t) - \chi_m y_{m-1_t}(\omega_t)\right] = hH_t(\omega) R_m\left(\vec{y}_{m-1_t}(\omega_t)\right), \quad \ldots(25)$$

where:

$$ R_m\left(\vec{y}_{m-1_t}(\omega_t)\right) = {}^{C}D_t^{1.4} y_{m-1_t}(\omega_t) + \omega_t^2\, {}^{C}D_t^{0.3} y_{m-1_t}(\omega_t) - (1-\chi_m) g_t(\omega_t), \quad \ldots(26)$$

So, letting $L = {}^{C}D_t^{1.4}$, $h = -1$ and $H_t(\omega_t) = 1$, and hence eq. (25) will take the form:

$$ {}^{C}D_t^{1.4}\left[y_{m_t}(\omega_t) - \chi_m y_{m-1_t}(\omega_t)\right] = -R_m\left(\vec{y}_{m-1_t}(\omega_t)\right). \quad \ldots(27)$$

Applying the Riemann-Liouville factional order integral ${}^{RL}I_t^{1.4}$ to the both sides of eq. (27) and using the initial approximate solution $y_{0_t}(\omega_t) = 0$, then the functions $y_{1_t}(\omega_t)$, $y_{2_t}(\omega_t)$, $\cdots$ may be evaluated one after one in order by solving the linear higher-order deformation equations:

$$ {}^{RL}I_t^{1.4}\, {}^{C}D_t^{1.4}\left[y_{m_t}(\omega_t) - \chi_m y_{m-1_t}(\omega_t)\right] = -{}^{RL}I_t^{1.4} R_m\left(\vec{y}_{m-1_t}(\omega_t)\right). \quad \ldots(28)$$

If $m = 1$, then:

$$ {}^{RL}I_t^{1.4}\, {}^{C}D_t^{1.4}\left[y_{1_t}(\omega_t) - \chi_1 y_{0_t}(\omega_t)\right] = -{}^{RL}I_t^{1.4} R_1\left(y_{0_t}(\omega_t)\right),$$

and since $y_{0_t}(\omega_t) = 0$, $\chi_1 = 0$, the last equation will take the form:

$$ \begin{aligned} y_{1_t}(\omega_t) &= -{}^{RL}I_t^{1.4} R_1\left(y_{0_t}(\omega_t)\right) \\ &= -{}^{RL}I_t^{1.4}\left[{}^{C}D_t^{1.4} y_{0_t}(\omega_t) + \omega_t^2\, {}^{C}D_t^{0.3} y_{0_t}(\omega_t) - (1-\chi_1) g_t(\omega_t)\right] \\ &= -\left[y_{0_t}(\omega_t) + \omega_t^2\, {}^{RL}I_t^{1.1} y_{0_t}(\omega_t) - {}^{RL}I_t^{1.4} g_t(\omega_t)\right] \\ &= {}^{RL}I_t^{1.4} g_t(\omega_t) = t^2 + \frac{\Gamma(3)}{\Gamma(4.1)} \omega_t^2 t^{3.1}. \end{aligned}$$

If $m = 2$, then:

$$ {}^{RL}I_t^{1.4}\, {}^{C}D_t^{1.4}\left[y_{2_t}(\omega_t) - \chi_2 y_{1_t}(\omega_t)\right] = -{}^{RL}I_t^{1.4} R_2\left(y_{1_t}(\omega_t)\right),$$

since $\chi_m = 1$, for all $m = 2,3,\cdots$ and hence:

$$ y_{2_t}(\omega_t) - y_{1_t}(\omega_t) = -{}^{RL}I_t^{1.4} R_2\left(y_{1_t}(\omega_t)\right),$$

where:

$$ R_2\left(y_{1_t}(\omega_t)\right) = {}^{C}D_t^{1.4} y_{1_t}(\omega_t) + \omega_t^2\, {}^{C}D_t^{0.3} y_{1_t}(\omega_t).$$

Thus:

$$ y_{2_t}(\omega_t) - y_{1_t}(\omega_t) = -y_{1_t}(\omega_t) - \omega_t^2\, {}^{RL}I_t^{1.1}\left[t^2 + \frac{\Gamma(3)}{\Gamma(4.1)} \omega_t^2 t^{3.1}\right],$$

and so, carrying the required calculations, getting:

$$ y_{2_t}(\omega_t) = -\frac{\Gamma(3)}{\Gamma(4.1)} \omega_t^2 t^{3.1} - \frac{\Gamma(3)}{\Gamma(5.2)} \omega_t^4 t^{4.2}.$$

Similarly, we can calculate $y_{3_t}(\omega_t)$, which is found to be:

$$ y_{3_t}(\omega_t) = \frac{\Gamma(3)}{\Gamma(5.2)} \omega_t^4 t^{4.2} + \frac{\Gamma(3)}{\Gamma(6.3)} \omega_t^6 t^{5.3},$$

and so on.

Using eq. (23), the approximate solution of eq. (24) using the HAM up to the third terms is given by Figure 2 with 1000 and 10000 generations of Brownian motion.

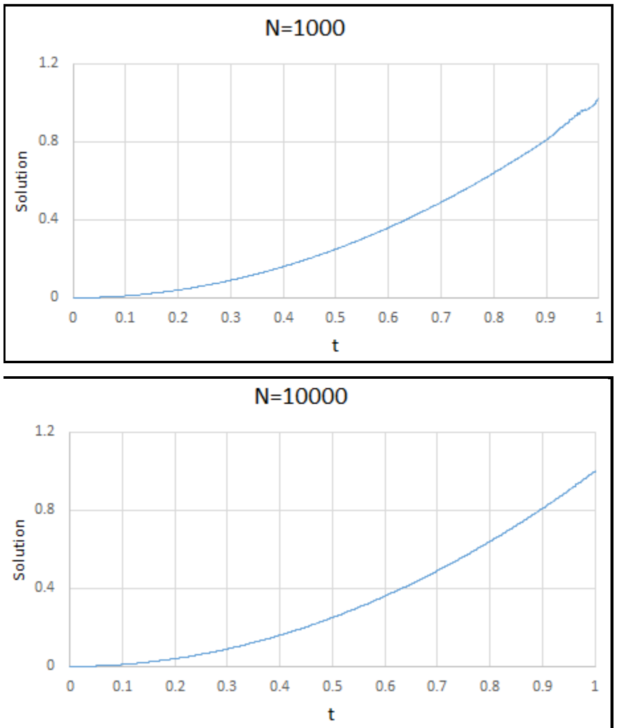


Figure 2. The approximate solution of Example 2 using the HAM for different number of Brownian motions 1000 and 10000, respectively.

Example 3. Consider the nonlinear fractional order random ODE:

$$ {}^{C}D_t^{0.5} y_t(\omega_t) + {}^{C}D_t^{0.3} y_t^2(\omega_t) = \frac{\omega_t}{\Gamma(1.5)} t^{0.5} + \frac{2\omega_t^2}{\Gamma(2.7)} t^{1.7},\ t \in [0,1], \quad \ldots(29)$$

with initial condition $y_0(\omega_0) = 0$.

To solve eq. (29) by means of HAM, we choose the initial approximation $y_{0_t}(\omega_t) = 0$ and letting:

$$N_t[y_t(\omega_t)] = D_t^{0.5} y_t(\omega_t) + {}^{C}D_t^{0.3} y_t^2(\omega_t) - \frac{\omega_t}{\Gamma(1.5)} t^{0.5} - \frac{2\omega_t^2}{\Gamma(2.7)} t^{1.7}.$$

According to eqs. (12)-(14) and with $L = {}^{C}D_t^{0.5}$, $h = -1$ and $H_t(\omega) = 1$, we have:

$$ {}^{C}D_t^{0.5}\left[y_{m_t}(\omega_t) - \chi_m y_{m-1_t}(\omega_t)\right] = -R_m\left(\vec{y}_{m-1_t}(\omega_t)\right),$$

and upon integrating both sides of the last equation with fractional order 0.5, implies to:

$$ {}^{RL}I_t^{0.5}\, {}^{C}D_t^{0.5}\left[y_{m_t}(\omega_t) - \chi_m y_{m-1_t}(\omega_t)\right] = -{}^{RL}I_t^{0.5} R_m\left(\vec{y}_{m-1_t}(\omega_t)\right),$$

where:

$$R_m\left(\vec{y}_{m-1_t}(\omega_t)\right) = {}^{C}D_t^{0.5} y_{m-1_t}(\omega_t) + \sum_{i=0}^{m-1} {}^{C}D_t^{0.3}\left(y_{i_t}(\omega_t) y_{m-1-i_t}(\omega_t)\right) - (1-\chi_1)\left(\frac{\omega_t}{\Gamma(1.5)} t^{0.5} + \frac{2\omega_t^2}{\Gamma(2.7)} t^{1.7}\right).$$

Thus, if $m = 1$:

$$ {}^{RL}I_t^{0.5}\, {}^{C}D_t^{0.5}\left[y_{1_t}(\omega_t) - \chi_1 y_{0_t}(\omega_t)\right] = -{}^{RL}I_t^{0.5} R_1\left(y_{0_t}(\omega_t)\right)$$

$$R_1\left(y_{0_t}(\omega_t)\right) = {}^{C}D_t^{0.5} y_{0_t}(\omega_t) + \sum_{i=0}^{0} {}^{C}D_t^{0.3}\left(y_{i_t}(\omega_t) y_{-i_t}(\omega_t)\right) - (1-\chi_1)\left(\frac{\omega_t}{\Gamma(1.5)} t^{0.5} + \frac{2\omega_t^2}{\Gamma(2.7)} t^{1.7}\right)$$

$$= {}^{C}D_t^{0.5} y_{0_t}(\omega_t) + {}^{C}D_t^{0.3}\left(y_{0_t}(\omega_t) y_{0_t}(\omega_t)\right) - \frac{\omega_t}{\Gamma(1.5)} t^{0.5} - \frac{2\omega_t^2}{\Gamma(2.7)} t^{1.7}$$

$$= -\frac{\omega_t}{\Gamma(1.5)} t^{0.5} - \frac{2\omega_t^2}{\Gamma(2.7)} t^{1.7}.$$

Hence:

$$y_{1_t}(\omega) = -{}^{RL}I_t^{0.5}\left(-\frac{\omega_t}{\Gamma(1.5)} t^{0.5} - \frac{2\omega_t^2}{\Gamma(2.7)} t^{1.7}\right)$$

$$= \frac{\omega_t \Gamma(1.5)}{\Gamma(1.7)} t + \frac{2\omega_t^2}{\Gamma(3.2)} t^{2.2},$$

if $m = 2$, then:

$$ {}^{RL}I_t^{0.5}\, {}^{C}D_t^{0.5}\left[y_{2_t}(\omega_t) - \chi_2 y_{1_t}(\omega_t)\right] = -{}^{RL}I_t^{0.5} R_2\left(y_{0_t}(\omega_t), y_{1_t}(\omega_t)\right) \quad \ldots(30)$$

$$R_2\left(y_{0_t}(\omega_t), y_{1_t}(\omega_t)\right) = {}^{C}D_t^{0.5} y_{1_t}(\omega_t) + \sum_{i=0}^{1} {}^{C}D_t^{0.3}\left(y_{i_t}(\omega_t) y_{1-i_t}(\omega_t)\right)$$

$$= {}^{C}D_t^{0.5} y_{1_t}(\omega_t) + 2\, {}^{C}D_t^{0.3}\left(y_{0_t}(\omega_t) y_{1_t}(\omega_t)\right)$$

$$= {}^{C}D_t^{0.5} y_{1_t}(\omega_t),$$

and thus from eq. (30):

$$y_{2_t}(\omega_t) = y_{1_t}(\omega_t) - {}^{RL}I_t^{0.5} R_2\left(y_{0_t}(\omega_t), y_{1_t}(\omega_t)\right)$$

$$= y_{1_t}(\omega_t) - {}^{RL}I_t^{0.5}\left({}^{C}D_t^{0.5} y_{1_t}(\omega_t)\right) = 0.$$

Similarly, if $m = 3$, then:

$$ {}^{RL}I_t^{0.5}\, {}^{C}D_t^{0.5}\left[y_{3_t}(\omega_t) - \chi_3 y_{2_t}(\omega_t)\right] = -{}^{RL}I_t^{0.5} R_3\left(y_{0_t}(\omega_t), y_{1_t}(\omega_t), y_{2_t}(\omega_t)\right)$$

$$R_3\left(y_{0_t}(\omega_t), y_{1_t}(\omega_t), y_{2_t}(\omega_t)\right) = {}^{C}D_t^{0.5} y_{2_t}(\omega_t) + \sum_{i=0}^{2} {}^{C}D_t^{0.3}\left(y_{i_t}(\omega_t) y_{2-i_t}(\omega_t)\right)$$

$$= {}^{C}D_t^{0.5} y_{2_t}(\omega_t) + {}^{C}D_t^{0.3}\left(y_{0_t}(\omega_t) y_{2_t}(\omega_t) + y_{1_t}^2(\omega_t) + y_{2_t}(\omega_t) y_{1_t}(\omega_t)\right)$$

$$= {}^{C}D_t^{0.3}\left(y_{1_t}^2(\omega_t)\right),$$

and so after carrying out some calculations:

$$y_{3_t}(\omega_t) = y_{2_t}(\omega_t) - {}^{RL}I_t^{0.5}\,{}^{C}D_t^{0.3}\left(y_{1_t}^2(\omega_t)\right)$$
$$= -{}^{RL}I_t^{0.2}\left(y_{1_t}^2(\omega_t)\right)$$
$$= -0.299812t^{4.9}\omega_t^4 - 0.572481t^{2.5}\omega_t^2 - 0.80902t^{3.7}\omega_t^3.$$

Similarly, if $m = 4$, we will get $y_{3_t}(\omega_t) = 0$, while if $m = 5$, implies:

$$y_{5_t}(\omega_t) = -1.26142t^{4.7}\omega_t^4 - 0.959935t^{5.9}\omega_t^5 - 0.558361t^{3.5}\omega_t^3 - 0.247373t^{7.1}\omega_t^6,$$

and so on.

Finally, the approximate solution is given by:

$$y_t(\omega_t) = y_{0_t}(\omega_t) + y_{1_t}(\omega_t) + y_{2_t}(\omega_t) + \cdots$$
$$= 0.825094t^{2.2}\omega_t^2 + 1.50699t^{4.9}\omega_t^4 - 0.572481t^{2.5}\omega_t^2 + 1.3201t^{6.1}\omega_t^5 + 0.328871t^{7.3}\omega_t^6 + 0.0327315t^{3.7}\omega_t^3 + 0.975335t\omega_t. \quad \ldots(31)$$

The approximate solution (31) of eq. (29) in terms of $y_{0_t}(\omega_t)$, $y_{1_t}(\omega_t)$, $y_{2_t}(\omega_t)$ and $y_{3_t}(\omega_t)$ with 1000 and 10000 generations of Brownian motion are drown respectively in Figure 3.

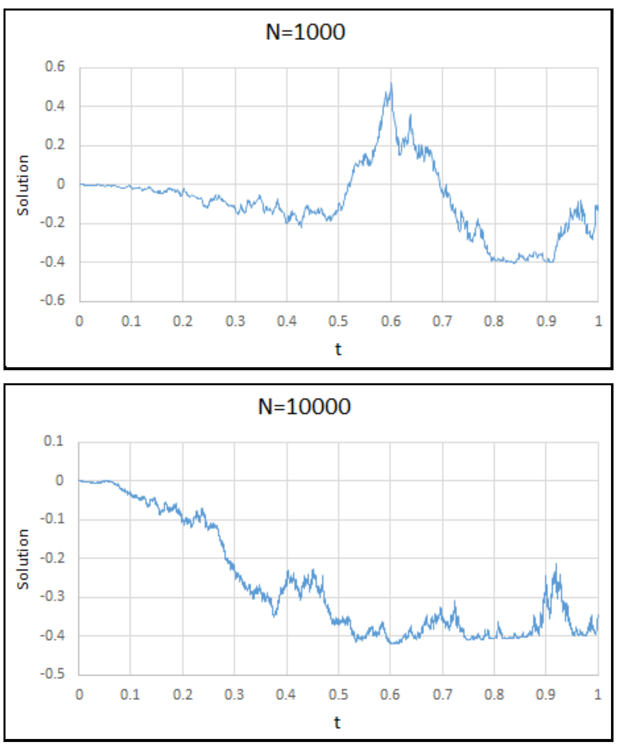


Figure 3. The approximate solution of Example 3 using the HAM for 1000 and 10000 number of Brownian motions, respectively.

7. Conclusions

The HAM was used in this study to derive approximate solutions to linear and nonlinear multi-fractional random ODEs. In conclusion, HAM produces accurate numerical results for such problems, and the convergence of the series solution can be controlled by selecting the appropriate auxiliary and homptopy parameters.

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طريقة التحليل المثيلة لحل المعادلات التفاضلية الاعتيادية العشوائية متعددية الرتب الكسورية

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المستخلص:

الهدف الرئيسي من هذا البحث هو تقديم معادلات تفاضلية عادية عشوائية ذات مشتقات كسرية متعددة واستخدام طريقة التحليل المثيلة لتقريب حل هذه المعادلات مع أجيال مختلفة من عملية وينر أو الحركة براون.

واحدة من أهم الطرق وأكثرها فعالية لحل المشكلات الرياضية المتنوعة باستخدام مؤثرات مختلفة كما في المعادلات التفاضلية الخطية وغير الخطية، والمعادلات التفاضلية العادية أو الجزئية، والمعادلات التكاملية، وما إلى ذلك، هي طريقة التحليل المثيلة.

الكلمات المفتاحية: المعادلات التفاضلية العشوائية، المشتقات الكسرية، طريقة التحليل المثيل، العمليات التصادفية.