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A numerical scheme for the solution of fractional integro-differential equations using the Adomian decomposition method.

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ABSTRACT

The aim of this paper is to apply the Adomian decomposition method for linear fractional differential equations. The definition of Riemann-Liouville for fractional derivative was used in this paper.

Introduction

The fractional order integro differential field is a rapidly growing field in both theory and applications, it is natural to study the numerical solution fractional integro differential equation [1, 2, 8]. We are concerned with providing a numerical scheme for the solution of fractional integro-differential equations of the general form:

$$\frac{d^\alpha y}{dx^\alpha} = g(x) + \int_a^b k\big(x,t,y(t)\big)\, dx \quad (1.1)$$

Subject to the initial conditions

$$y^{(i)}(0) = c_i, i = 0,1,2,\cdots, m-1 \quad (1.2)$$

where $m-1 < \alpha \leq m$ and $m \in \mathbb{N}$.

In this paper we consider

$$\frac{d^\alpha y}{dx^\alpha} = g(x) + \int_a^b k(x,t)y(t)\, dt \quad (1.3)$$

subject to the initial conditions

$$y(0) = c_0, \quad (1.4)$$

where $0 < \alpha \leq 1$

The Adomian decomposition method [3] will be applied for computing solutions to the fractional integro-differential equations (1.1)-(1.4). The Adomian decomposition method has many advantages over the classical techniques mainly, it avoids discretization and provide an efficient numerical solution with high accuracy and minimal calculations [5],[6].

We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In section 3 we extend the application of the decomposition method to construct our numerical solutions for the integro differential equations (1.1)-(1.2). While the linear case were discussed in section 4.

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Preliminaries and notations

This section is devoted to a description of the operational properties of the purpose of acquainting with sufficient fractional calculus theory, to enable us to follow the solutions for the problem given in this paper. Many definitions and studies of fractional calculus have been proposed in last two centuries.

Definition 2.1 Let $\alpha \in \mathbb{R}^+$. The operator J_a^α, defined on the usual Lebesque space $L_1[a,b]$ by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

$$J_a^0 f(x) = f(x)$$

For $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order α.

Properties of the operator J_a^α can be found in [7], we mention the following :

For $f \in L_1[a,b], \alpha, \beta \geq 0$ and $\gamma > -1$

1. $J_a^\alpha f(x)$ exists for almost every $x \in [a,b]$.
2. $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$.
3. $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$.
4. $J_a^\alpha (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a)^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations.

Therefore we shall introduce a modified fractional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [4, 9].

Definition 2.2 The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^\alpha f(x)$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f(t)dt \quad (2.1)$$

for $m-1 < \alpha \le m$ and $m \in \mathbb{N}, x > 0$.

Also, we need here two of its basic properties.

Lemma 2.1

If $m-1 < \alpha \le m$ and $f \in L_1[a,b]$, then

$$D_a^\alpha J_a^\alpha f(x) = f(x).$$

and

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+)\frac{(x-a)^i}{i!}, x > 0.$$

In this case of $f(x) = (x-a)^\gamma$ we have

$$D_a^\alpha f(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha-\gamma+1)}(x-\alpha)^{\gamma-\alpha}.$$

3 Analysis of the numerical method

The decomposition method [3,10,11] requires that the fractional integro-differential equation (1.1) be expressed in term of operator form as:

$$D_a^\alpha y(x) + Ly(x) + Ny(x) = g(x), \quad (3.1)$$

where L is a linear operator which may include other fractional derivatives of order less than α, N is a nonlinear which also may include other fractional derivatives of order less than α and the fractional operator D_a^α is defined as in equation (2.1) denoted by $D_a^\alpha = \frac{d^\alpha}{dx^\alpha}$.

The method is based on applying the operator $-J^\alpha = J_0^\alpha$, the inverse of the operator D_x^α , formally to the expression

$$D_x^\alpha y(x) = g(x) - Ly(x) - Ny(x). \quad (3.2)$$

Following Adomian, we write

$Ny = \sum_{k=0}^\infty A_k$ and

$$y(x) = \sum_{k=0}^\infty y_k(x), \quad (3.3)$$

where A_k are so called Adomian polynomials.

Now operating with J^α on both sides of (3.2) yields

$$y(x) = \sum_{i=0}^{m-1} y^{(i)}(0^+)\frac{x^i}{i!} + J^\alpha g(x) - J^\alpha Ly(x) - J^\alpha Ny(x). \quad (3.4)$$

Inserting (3.3) into (3.4), we define

$$y_0(x) = \sum_{i=0}^{m-1} y^{(i)}(0^+)\frac{x^i}{i!} + J^\alpha g(x).$$

where $g(x)$ is the source term in (1.1).

Now we define successively

$$y_1 = -J^\alpha L y_0 - J^\alpha A_0$$
$$y_2 = -J^\alpha L y_1 - J^\alpha A_1$$
$$y_3 = -J^\alpha L y_2 - J^\alpha A_2$$

and so on.

As a result, the series solution is given by

$$y(x) = \sum_{k=0}^\infty y_k(x). \quad (3.5)$$

Define the γ-term approximation solution as

$$\phi_\gamma = \sum_{k=0}^{\gamma-1} y_k(x), \quad (3.6)$$

And the exact solution *y(x)* is given by

$$y(x) = \lim_{\gamma\to\infty} \phi_\gamma \quad (3.7)$$

The Adomian polynomial can be calculated for all forms of nonlinearity $\phi(y)$ according to specific algorithms constructed by Adomian. The general form of formula for A_k Adomian polynomials as

$$A_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k}\phi\left(\sum_{i=0}^\infty \lambda^i y_i\right)\right]_{\lambda=0}.$$

This formula is easy to compute by using Mathematica software or by setting a computer code to get as many polynomials as we need in the calculations of the numerical as well as explicit solution. The first few terms of the Adomian polynomials for the nonlinear function $Ny = \phi(y)$ are derived as follows

$$A_0 = \phi(y_0),$$
$$A_1 = y_1\phi^{(1)}(y_0),$$
$$A_2 = y_2\phi^{(1)}(y_0) + \frac{y_1^2}{2!}\phi^{(2)}(y_0),$$
$$A_3 = y_3\phi^{(1)}(y_0) + y_1y_2\phi^{(2)}(y_0) + \frac{y_1^3}{3!}\phi^{(3)}(y_0)$$

.

Linear fractional integro differential equations

To use the decomposition method we need to rewrite the fractional integro-differential equation (1.3) in term of operator form as:

$$D_a^\alpha y(x) + Ly(x) + Ny(x) = g(x), \quad (4.1)$$

where L is a linear operator which may include other fractional derivatives of order less than α, N is a nonlinear which also may include other fractional derivatives of order less than α and the fractional operator D_a^α is defined as in equation (2.1) denoted by $D_a^\alpha = \frac{d^\alpha}{dx^\alpha}$.

Applying the operator $J^\alpha = J_0^\alpha$, formally to the expression

$$D_x^\alpha y(x) = g(x) - Ly(x). \quad (4.2)$$

Following Adomian, we write

$$y(x) = \sum_{k=0}^\infty y_k(x), \quad (4.3)$$

Now operating with J^α on both sides of (4.2) yields

$y(x) = y(0) + J^{\alpha} g(x) - J^{\alpha} L y(x).$ (4.4)

Inserting (4.3) into (4.4), we define

$y_0(x) = y(0) + J^{\alpha} g(x),$

where $g(x)$ is the source term in (1.3).

Now we define successively

$$y_1 = -J^{\alpha} L y_0 = -J^{\alpha}\left[\int_a^b k(x,t) y_0(t)\,dt\right]$$

$$y_2 = -J^{\alpha} L y_1 = -J^{\alpha}\left[\int_a^b k(x,t) y_1(t)\,dt\right]$$

$$y_3 = -J^{\alpha} L y_2 = -J^{\alpha}\left[\int_a^b k(x,t) y_2(t)\,dt\right]$$

$$\vdots$$

and so on.

As a result, the series solution is given by

$y(x) = \sum_{k=0}^{\infty} y_k(x).$ (4.5)

Define the γ-term approximation solution as

$\phi_{\gamma} = \sum_{k=0}^{\gamma-1} y_k(x),$ (4.6)

And the exact solution *y(x)* is given by

$y(x) = \lim_{\gamma\to\infty} \phi_{\gamma}.$ (4.7)

References

[1] A. Arikoglu, I. Ozkol, "Solution of boundary value problem for integro-differential equations by using differential transform method, Appl. Math. Comp. in press.

[2] F. Minardi, (1997). Fractional calculus: " Some basic problems in continuum and statistical mechanics",A. Carpinter and F. Minardi (eds), *Springer-Verlag*, New York, 291-348.

[3] G. Adomian,(1994). Solving Frontier Problems of physics: The Decomposition method, *Kluwer, Academic Publisher*. Boston, USA

[4] I. Poldubny, (2002). Geometric and physical interpretation of fractional integration and fractional differentiation, *Fractal calculus Appl. Anal.*, 5, 367-386.

[5] K. Abbaoui, Y. Charruault (1996). "New ideas for proving convergence of decomposition methods", *Comp. Math. Applications*, 29(7) 103-108.

[6] K. Abbaoui, Y. Charruault, (1996). " Convergence of Adomian's method applied to differential equations", *Comp. Math. Applications*, 28(5) 103-106.

[7] K. B. Oldham, J. Spanier,(1974)."The fractional calculus", *Academic Press*, *New York*, USA.

[8] K. Diethelm, N. J. Ford, A. D. Freed, Yu Luchko, "Algorithms for the fractional calculus, aselection of numerical methods", Comp. method in appl. Mech. And Eng. In press.

[9] M. Caputo (1967). Linear models of dissipation whose Q is almost frequency independent. Part II, *J. Roy. Astral. Soc.*, 13, 529-539.

[10] Y. Charruault, (1989). Convergence of Adomian's method, *Kyberenetas*, 18 31-38.

[11] Y. Charruault, G. Adomian (1993), decomposition method a new proof of convergence, *Math. Comp. Modeling*, 18103-106.

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