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New iterative technique for computing Fourier transforms.

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ABSTRACT

The Fourier transformations have stimulated many amounts of articles in recent years. they arise in the fields of engineering, control systems, and technology like analyzing signals in electronic circuits, radio circuits, cell phones, image processing, and in solutions to heat transfer equations, Airy equations, Telegraph equations, Duffing equations, Wave equations, Fisher equations, Laplace equation, etc. In this paper, a new iterative method called Adomian Decomposition Method (ADM) is implemented to obtain the Fourier transform of functions by solving a linear ordinary differential equation of first order. This method focuses on finding Fourier transforms by knowing the series resulting from Adomian polynomials. Five famous examples are presented to test the effectiveness and validity of this technique. The results indicate that the accuracy of this method is fully in agreement with the classical method. Furthermore, when applying the Adomian decomposition method, we noticed that it provides accurate results and does not require a lot of time and effort to obtain Fourier transforms of the functions because it does not require a large number of iterations.

Introduction

The topic of integral transformations is one of the important topics used in solving many physical and engineering problems [4,5,7,9,10,11,13]. One of these transformations is the Fourier Transform, this transform decomposes complex signals and converts them into sinusoidal components, these signals can be expressed by the frequency of waves [14,15].

Definition 1. [2] The Fourier Transform of $g(v)$ denoted by $\mathcal{F}$ is given by

$$\mathcal{F}[g(v)] = \int_{v=-\infty}^{v=\infty} g(v)\, e^{-iwv}\ dv = \hat{g}(w)$$

Definition 2. [2] The inverse Fourier transform of $\hat{g}(w)$ is given by

$$\mathcal{F}^{-1}[\hat{g}(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(w)\, e^{iwv}\ dw = g(v)$$

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Definition 3. [16] The Dirac delta distribution is limit for $\varepsilon \to 0$ function defined by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & 0 < t < \varepsilon \\ 0, & t < 0 \\ 0, & t > \varepsilon \end{cases}$$

That is $\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$.

Some properties of the Dirac Delta distribution are as follows [8, 16]:

I. $\delta(w) = \begin{Bmatrix} \infty, & w = 0 \\ 0, & w \neq 0 \end{Bmatrix}$.

II. $\int_{-\infty}^{\infty} e^{-i(w\pm b)v}\ dv = 2\pi\delta(w \pm b)$, for $b \in \mathbb{R}$.

Recently, Fourier transforms of functions have been calculated using different methods. Düz. et al [1] have implemented the Differential transformation method for computing Fourier transforms. Issa. et al [2] have solved Fourier transforms by using the variational iteration method. In this article, we will introduce another technique (Adomian decomposition method) for calculating Fourier transforms of functions with linear ODEs of the first order as shown

$$\gamma' - iw\gamma = ig(v), w \in \mathbb{C}, \qquad \gamma(0) = 0 \quad (1)$$

and we will provide some important examples to demonstrate the efficiency of the proposed method.

Table 1: The Fourier transforms of functions

$g(v)$	$\mathcal{F}[g(v)]$
1	$2\pi\delta(w)$
v^m	$2\pi i^m \delta^{(m)}(w)$
e^{av}	$2\pi\delta(w+ai)$
$\sin av$	$\frac{\pi}{i}(\delta(w-a)-\delta(w+a))$
$\cos av$	$\pi(\delta(w-a)+\delta(w+a))$

Applying Adomian decomposition method to Equation (1) :

Now we let apply the Adomian decomposition method [3,12] to equation (1)

$$\gamma' - iw\gamma = ig(v)$$

$$L\gamma = iw\gamma + ig(v), \qquad L = \frac{d}{dv}$$

$$L^{-1}L\gamma = L^{-1}(iw\gamma) + iL^{-1}(g(v)), \quad L^{-1} = \int .\, dv$$

$$\gamma_{n+1} = iwL^{-1}(\gamma_n)$$

$$\gamma_0 = \gamma(0) + iL^{-1}(g(v))$$

$$\gamma_1 = iwL^{-1}(\gamma_0)$$

$$\gamma_2 = iwL^{-1}(\gamma_1)$$

$$\gamma_3 = iwL^{-1}(\gamma_2)$$

$$\vdots$$

As usual in Adomian decomposition method the solution of Eq. (1) is considered to be as the sum of a series:

$$\gamma = \sum_{n=0}^{\infty} \gamma_n$$

Theorem 3.1 : Consider the linear ordinary differential equations of first order as shown

$$\gamma' - iw\gamma = ig(v), w \in \mathbb{C}, \qquad \gamma(0) = 0 \ \ (1)$$

Moreover, let $g(v)$ be an analytic function, then the Fourier transform of $g(v)$ is

$$\mathcal{F}[g(v)] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty} \quad (2).$$

Where γ_n's is obtained with the Adomian decomposition method from equation (1).

Proof : we let solve the equation (1)

$$\gamma' - iw\gamma = ig(v), \qquad \lambda = e^{\int -iw\, dv} = e^{-iwv}$$

$$\left(\gamma e^{-iwv}\right)' = ig(v)e^{-iwv}$$

Integrating both sides from $-\infty$ to ∞ with respect to v, we get the relation between the solution of equation (1) and Fourier Transform of $g(v)$ as

$$\lim_{a\to\infty}\lim_{b\to\infty} \gamma e^{-iwv}\Big|_b^a = i\int_{-\infty}^{\infty} g(v)e^{-iwv}\, dv = i\mathcal{F}[g(v)]$$

Therefore,

$$\mathcal{F}[g(v)] = \left[\frac{\gamma e^{-iwv}}{i}\right]\Big|_{-\infty}^{\infty} = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty}.$$

Examples :

In this section, we will use the Adomian decomposition method to get the Fourier Transforms for some important functions

Example 1. Let $g(v) = 1$, and by using equation (2), we have

$$\mathcal{F}[1] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty} \ (3)$$

Now we find some of γ_n's

$$\gamma_0 = \gamma(0) + iL^{-1}(1), \qquad \gamma_{n+1} = iwL^{-1}(\gamma_n)$$

$$\gamma_0 = iv$$

$$\gamma_1 = iw\frac{iv^2}{2} = \frac{i^2wv^2}{2}$$

$$\gamma_2 = \frac{i^2w^2iv^3}{6} = \frac{i^3w^2v^3}{6}$$

$$\gamma_3 = \frac{i^4w^3v^4}{24}$$

$$\gamma_4 = \frac{i^5w^4v^5}{120}$$

$$\vdots$$

$$\gamma_n = \frac{i^{n+1}w^nv^{n+1}}{(n+1)!}$$

Finally, we get the Fourier transform of 1 by substituting the previous equations in (3)

$$\mathcal{F}[1] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-iwv}}{i}\left[iv + \frac{i^2wv^2}{2} + \frac{i^3w^2v^3}{6} + \frac{i^4w^3v^4}{24} + \cdots\right]\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-iwv}}{i}\left[\frac{e^{iwv}-1}{w}\right]\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{1-e^{-iwv}}{iw}\right]\Big|_{v=-\infty}^{v=\infty} = \left[\frac{e^{-iwv}}{-iw}\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \int_{-\infty}^{\infty} e^{-iwv}\, dv = 2\pi\delta(w).$$

Example 2. Let $g(v) = v^m$, and by using equation (2), we have

$$\mathcal{F}[v^m] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty} \quad (4)$$

Now we find some of γ_n's

$$\gamma_0 = \gamma(0) + iL^{-1}(v^m)$$

$$\gamma_0 = i\frac{v^{m+1}}{(m+1)}$$

$$\gamma_{n+1} = iwL^{-1}(\gamma_n)$$

$$\gamma_1 = iw\frac{iv^{m+2}}{(m+1)(m+2)} = \frac{i^2wv^{m+2}}{(m+1)(m+2)}$$

$$\gamma_2 = \frac{i^3w^2v^{m+3}}{(m+1)(m+2)(m+3)}$$

$$\gamma_3 = \frac{i^4w^3v^{m+4}}{(m+1)(m+2)(m+3)(m+4)}$$

$$\gamma_4 = \frac{i^5w^4v^{m+5}}{(m+1)(m+2)(m+3)(m+4)(m+5)} = \frac{i^5w^4\, m!\, v^{m+5}}{(m+5)!}$$

$$\vdots$$

$$\gamma_n = \frac{i^{n+1}w^n\, m!\, v^{m+n+1}}{(m+n+1)!}$$

Finally, we get the Fourier transform of v^m by substituting the previous equations in (4)

$$\mathcal{F}[v^m] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-iwv}}{i}\left[i\frac{v^{m+1}}{(m+1)} + \frac{i^2wv^{m+2}}{(m+1)(m+2)} + \frac{i^3w^2v^{m+3}}{(m+1)(m+2)(m+3)} + \frac{i^4w^3v^{m+4}}{(m+1)(m+2)(m+3)(m+4)} + \frac{i^5w^4\, m!\, v^{m+5}}{(m+5)!} + \cdots\right]\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-iwv}}{i}\left[i\frac{m!\, v^{m+1}}{(m+1)!} + \frac{i^2w\, m!\, v^{m+2}}{(m+2)!} + \frac{i^3w^2m!\, v^{m+3}}{(m+3)!} + \frac{i^4w^3m!\, v^{m+4}}{(m+4)!} + \frac{i^5w^4\, m!\, v^{m+5}}{(m+5)!} + \cdots\right]\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-iwv}}{i}\left[\frac{i\, m!}{(iw)^{m+1}}\right]\left[e^{iwv} - 1 - iwv - \frac{(iwv)^2}{2!} - \cdots - \frac{(iwv)^m}{m!}\right]\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \frac{2\pi}{(-1)^m i^m}\left[\frac{(-1)^{m+1}i^m e^{-iwv}}{2\pi}\left(\frac{v^m}{iw} + \frac{m\, v^{m-1}}{(iw)^2} + \cdots + \frac{m!}{(iw)^{m+1}}\right)\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= 2\pi i^m\delta^{(m)}(w).$$

Example 3. Let $g(v) = e^{av}$, and by using equation (2), we have

$$\mathcal{F}[e^{av}] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty} \quad (5)$$

Now we find some of γ_n's

$$\gamma_0 = \gamma(0) + iL^{-1}(e^{av}) = \frac{i}{a}e^{av}$$

$$\gamma_{n+1} = iwL^{-1}(\gamma_n)$$

$$\gamma_1 = iwL^{-1}\left(\frac{i}{a}e^{av}\right) = \frac{i^2w}{a^2}e^{av}$$

$$\gamma_2 = iwL^{-1}\left(\frac{i^2w}{a^2}e^{av}\right) = \frac{i^3w^2}{a^3}e^{av}$$

$$\gamma_3 = \frac{i^4w^3}{a^4}e^{av}$$

$$\gamma_4 = \frac{i^5w^4}{a^5}e^{av}$$

$$\vdots$$

Finally, we get the Fourier transform of e^{av} by substituting the previous equations in (5)

$$\mathcal{F}[e^{av}] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-iwv}}{i}\left[\left(\frac{i}{a} + \frac{i^2w}{a^2} + \frac{i^3w^2}{a^3} + \frac{i^4w^3}{a^4} + \frac{i^5w^4}{a^5} + \cdots\right)e^{av}\right]\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-i(w+ai)v}}{-i(w+ai)}\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \int_{-\infty}^{\infty} e^{-i(w+ai)v}\, dv = 2\pi\delta(w+ai).$$

Example 4. Let $g(v) = \sin av$, and by using equation (2), we have

$$\mathcal{F}[\sin av] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty} \quad (6)$$

Now we find some of γ_n's

$$\gamma_0 = \gamma(0) + iL^{-1}(\sin av) = iL^{-1}\left(\frac{1}{2i}\left(e^{iav} - e^{-iav}\right)\right)$$

$$= \frac{1}{2}L^{-1}\left(e^{iav} - e^{-iav}\right) = \frac{1}{2ia}\left(e^{iav} + e^{-iav}\right)$$

$$\gamma_{n+1} = iwL^{-1}(\gamma_n)$$

$$\gamma_1 = iwL^{-1}\left(\frac{1}{2ia}\left(e^{iav} + e^{-iav}\right)\right)$$

$$= \frac{w}{2ia^2}\left(e^{iav} - e^{-iav}\right)$$

$$\gamma_2 = iwL^{-1}\left(\frac{w}{2ia^2}\left(e^{iav} - e^{-iav}\right)\right)$$

$$= \frac{w^2}{2ia^3}\left(e^{iav} + e^{-iav}\right)$$

$$\gamma_3 = \frac{w^3}{2ia^4}\left(e^{iav} - e^{-iav}\right)$$

$$\gamma_4 = \frac{w^4}{2ia^5}\left(e^{iav} + e^{-iav}\right)$$

$$\vdots$$

Finally, we get the Fourier transform of $\sin av$ by substituting the previous equations in (6)

$$\mathcal{F}[\sin av] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-iwv}}{i}\left[\left(\frac{1}{2ia} + \frac{w^2}{2ia^3} + \frac{w^4}{2ia^5} + \cdots\right)\left(e^{iav} + e^{-iav}\right) + \left(\frac{w}{2ia^2} + \frac{w^3}{2ia^4} + \frac{w^5}{2ia^6} + \cdots\right)\left(e^{iav} - e^{-iav}\right)\right]\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \frac{1}{2i}\left[\frac{e^{-i(w-a)v}}{i(a-w)} + \frac{e^{-i(w+a)v}}{i(a+w)}\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty}\left(e^{-i(w-a)v} - e^{-i(w+a)v}\right)dv$$

$$= \frac{\pi}{i}\left(\delta(w-a) - \delta(w+a)\right).$$

Example 5. Let $g(v) = rect(v) = \begin{cases}\frac{1}{b}, & -\frac{b}{2} \le v \le \frac{b}{2} \\ 0, & otherwise\end{cases}$, and by using equation (2), we have

$$\mathcal{F}[rect(v)] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty} \quad (7)$$

Now we find some of γ_n's

$$\gamma_0 = \gamma(0) + iL^{-1}\left(\frac{1}{b}\right), \qquad \gamma_{n+1} = iwL^{-1}(\gamma_n)$$

$$\gamma_0 = \frac{i}{b}v$$

$$\gamma_1 = iw\,\frac{iv^2}{2b} = \frac{i^2wv^2}{2b}$$

$$\gamma_2 = \frac{i^2w^2iv^3}{6b} = \frac{i^3w^2v^3}{6b}$$

$$\gamma_3 = \frac{i^4w^3v^4}{24b}$$

$$\gamma_4 = \frac{i^5w^4v^5}{120b}$$

$$\vdots$$

$$\gamma_n = \frac{i^{n+1}w^nv^{n+1}}{b(n+1)!}$$

Finally, we get the Fourier transform of $rect(v)$ by substituting the previous equations in (7)

$$\mathcal{F}[rect(v)] = \left[\frac{e^{-iwv}}{i}\sum_{n=0}^{\infty}\gamma_n\right]\Big|_{v=-\infty}^{v=\infty}$$

$$= \left[\frac{e^{-iwv}}{i}\left[\frac{i}{b}v + \frac{i^2wv^2}{2b} + \frac{i^3w^2v^3}{6b} + \frac{i^4w^3v^4}{24b} + \cdots\right]\right]\Big|_{v=-\frac{b}{2}}^{v=\frac{b}{2}}$$

$$= \left[\frac{e^{-iwv}}{ib}\left[\frac{e^{iwv}-1}{w}\right]\right]\Big|_{v=-\frac{b}{2}}^{v=\frac{b}{2}} = \left[\frac{1-e^{-iwv}}{ibw}\right]\Big|_{v=-\frac{b}{2}}^{v=\frac{b}{2}}$$

$$= \frac{\sin\left(\frac{bw}{2}\right)}{\frac{bw}{2}} = \text{sinc}\left(\frac{bw}{2\pi}\right).$$

The formula of sinc function in [6].

Conclusion

In this paper, we have dealt with the Fourier transform and important definitions and properties of it. Furthermore, the application of the Adomian decomposition method to calculate the Fourier transform of functions has been demonstrated, which are important transforms in applied mathematics.

References

[1] Düz, M., Issa, A., & Avezov, S. (2022). A new computational technique for Fourier transforms by using the Differential transformation method, *Bulletin of International Mathematical Virtual Institute*, 12(2), 287-295.

[2] Issa, A., & Düz, M. (2022). Different computational approach for Fourier transforms by using variational iteration method, *Journal of New Results in Science*, 11(3), 190-198.

[3] Chen, W., & Lu, Z. (2004). An algorithm for Adomian decomposition method, *Applied Mathematics and Computation*, 159(1), 221-235.

[4] Abbasbandy, S. (2006). Application of He's homotopy perturbation method for Laplace transform, *Chaos, Solitons & Fractals*, 30(5), 1206-1212.

[5] Babolian, E., Biazar, J., & Vahidi, A. (2004). A new computational method for Laplace transforms by decomposition method, *Applied Mathematics and Computation*, 150(3), 841-846.

[6] Issa, A., & Al Horani, M. (2023). Sinc collocation method for solving linear systems of Fredholm Volterra integro–differential equations of high order with variable coefficients, *Results in Nonlinear Analysis*, 6(1), 49-58.

[7] Osgood, B. (2009). The Fourier transform and its applications, *Lecture notes for EE*, 261, 20.

[8] Wheeler, N. (1997). Simplified production of Dirac delta function identities, *Reed College*.

[9] Adomian, G. (1988). Nonlinear stochastic systems theory and applications to physics, *Springer Science & Business Media*, 46.

[10] Adomian, G. (2013). Solving frontier problems of physics: the decomposition method, *Springer Science & Business Media*, 60.

[11] Haldar, K., & Datta, B.K. (1996). Integrations by asymptotic decomposition, *Appl. Math. Lett*, 9(2) 81–83.

[12] Cherruault, Y. (1989). Convergence of Adomian's method, *Kybernets*, 18(2), 31–39.

[13] Cherruault, Y. (1992). Some new results for convergence of Adomians method applied to integral equations, *Math. Comput. Modeling*, 16(2), 85–93.

[14] Bracewell, R.N. (1989). The fourier transform, *Scientific American*, 260(6), 86-95.

[15] Ernst, R.R., & Anderson, W.A. (1966). Application of Fourier transform spectroscopy to magnetic resonance, *Review of Scientific Instruments*, 37(1), 93-102.

[16] Avezov, S., Düz, M., & Issa, A. (2023). Solutions to Differential-Differential Difference Equations with Variable Coefficients by Using Fourier Transform Method, Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi, 18(3), 259-267.

تقنية تكرارية جديدة لحساب تحويلات فورييه

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الخلاصة:

لقد حفزت تحويلات فورييه العديد من المقالات في السنوات الأخيرة. تنشأ في مجالات الهندسة وأنظمة التحكم والتكنولوجيا مثل تحليل الإشارات في الدوائر الإلكترونية، ودوائر الراديو، والهواتف المحمولة، ومعالجة الصور، وفي حلول معادلات نقل الحرارة، معادلات إيري، معادلات التلغراف، معادلات دافينغ، معادلات الموجة، معادلات فيشر، معادلة لابلاس، الخ. في هذا البحث، تم تطبيق طريقة تكرارية جديدة تسمى طريقة تحليل أدوميان (ADM) للحصول على تحويل فورييه للدوال عن طريق حل معادلة تفاضلية خطية عادية من الدرجة الأولى. تركز هذه الطريقة على إيجاد تحويلات فورييه من خلال معرفة المتسلسلة الناتجة من كثيرات حدود أدوميان. تم عرض خمسة أمثلة مشهورة لاختبار فعالية وصلاحية هذه التقنية. وتشير النتائج إلى أن دقة هذه الطريقة تتفق تماما مع الطريقة الكلاسيكية. علاوة على ذلك، عند تطبيق طريقة تحليل أدوميان، لاحظنا أنها توفر نتائج دقيقة ولا تتطلب الكثير من الوقت والجهد للحصول على تحويلات فورييه للدوال لأنها لا تتطلب عددا كبيرا من التكرارات.

الكلمات المفتاحية : طريقة تحليل أدوميان ، تحويل فورييه ، معادلات تفاضلية.