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ON DIFFERENTIAL IDEALS OF DIFFERENTIAL RINGS

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ABSTRACT

In this paper we introduce two operators denoted by $(\ )_{(n)}$ and $(\ )_u$ of a differential ring constructed from a subset of a differential ring. We shall also discuss the relationship between these operators and the differential ideals in differential rings, and Keigher differential ring.

Introduction

Rings considered in this paper are all commutative with unity. The 0 ring has $1=0$. Also, all differential rings are ordinary , i.e., posses a single derivation .Recall that by a derivation of a ring R we means any additive map $\delta: R \to R$ satisfying $\delta(ab)=\delta(a)b+a\delta(b)$ for every $a,b \in R$. A differential ring R is a ring with a derivation δ. If R is a differential ring and $a \in R$, then $a^{(n)}$ denotes the n th derivative of a. A subset A of R is called differential if $\delta(A) \subseteq A$. For any subset A of R , the set $A_\delta = \{a \in A : \delta(a) \in A\}$ is called the differential of A.

Let R be a differential ring and let A be a subset of R . We define a subset , denoted by $A_{(n)}$,of R by $A_{(n)} = \{a : a^{(n)} \in A,$ for all $n \geq 0\}$. The following two theorems give some of the properties of $A_{(n)}$.

Theorem 1.1. *Let R be a differential ring . Then (1) If $A \subset R$, then $A_{(n)} \subset A$ and $(A_{(n)})_{(n)} = A_{(n)}$.*

(2) If $A \subset R$, then $A_{(n)} = A$ iff A is differential subset of R .

(3) If A,B are subsets of R with $A \subset B$, then $A_{(n)} \subset B_{(n)}$.

(4) If $\{A_\alpha\}_{\alpha \in I}$ is a family of subsets of R , then $(\bigcap_{\alpha\in I} A_\alpha)_{(n)} = \bigcap_{\alpha\in I}(A_\alpha)_{(n)}$ and $(\bigcup_{\alpha\in I} A_\alpha)_{(n)} \supset \bigcup_{\alpha\in I}(A_\alpha)_{(n)}$.

(5) If A,B are subsets of R , then $(A+B)_{(n)} \supset A_{(n)} + B_{(n)}$ and $(A.B)_n \supset A_{(n)}.B_{(n)}$.

Theorem 1.2. *Let R and S be differential rings and let $\varphi: R \to S$ be differential ring homomorphism such that $\varphi(1)=1$.. If A is a subset of R and B is a subset of S , then $\varphi(A_{(n)}) = (\varphi(A))_{(n)}$ and $\varphi^{-1}(B_{(n)}) = (\varphi^{-1}(B))_{(n)}$.*

The proof of these theorems is elementary and follows immediately from the definitions .

From theorems 1.1 and 1.2, we see that for any subset A of a differential ring R , $A_{(n)}$ is a differential subset . Also, the union and the intersection of any family of differential subsets is again a differential subset , and finite sums and products of differential subsets are differential subsets . Moreover , direct and inverse images of differential subsets under a differential ring homomorphism are differential.

Let A be a subset of a differential ring R .We define a subset , denoted by A_u , of R by $A_u = \{a \in A : \exists b \in A$ such that $ab = 1\}$. Hence , if A is a subring of R , A_u is the set of units in A.

Theorem 1.3. *Let R be a differential ring and S a subring of R . Then $(S_{(n)})_u = S_{(n)} \cap S_u$.*

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Proof. It is clear that $(S_{(n)})_u \subset S_{(n)} \cap S_u$, so let $a \in S$ be such that $a^{(n)} \in S$ for all $n \geq 0$, and suppose that $ab = 1$ for some $b \in S$. We want to show that $b^{(n)} \in S$ for all $n \geq 0$. We may assume $n \geq 1$ and that for each $k < n, b^{(k)} \in S$. Then by Leibnitz's rule [6] we have

$$0 = (ab)^{(n)} = ab^{(n)} + \sum_{k=1}^{n} \binom{n}{k} a^{(k)} b^{(n-k)} ,$$

So that

$$b^{(n)} = -b\left(\sum_{k=1}^{n} \binom{n}{k} a^{(k)} b^{(n-k)}\right) \in S$$

Hence , $(S_{(n)})_u = S_{(n)} \cap S_u$.

2. DIFFERENTIAL IDEALS AND KEIGHER RINGS

Theorem 2.1. *Let R be a differential ring and let A be a subset of R , then*

(1) If A is a subring of R , then $A_{(n)}$ is a subring of R.

(2) If A is an ideal of R , then $A_{(n)}$ is an ideal of R.

Proof. The proof of part (1) follows immediately from the definition. To prove part (2) ,suppose $x \in R$ and $a \in A_{(n)}$. Then by Leibentiz's rule [6] we have

$$(x\,a)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} a^{(n-k)}$$

Since every $a^{(n-k)} \in A$ and A is an ideal of $R, (x\,a)^{(n)} \in A$ and hence $x\,a \in A_{(n)}$. So that , $A_{(n)}$ is an ideal of R.

Recall that by a Ritt algebra [5] we means any differential ring which contains the rational numbers. Also, if I is an ideal of a differential ring R, the set $r(I) = \{a \in R : a^n \in I$ for some $n \in \mathbb{Z}^+\}$ is called the radical of I . An ideal I of R is called a radical ideal if $r(I) = I$.

Theorem 2.2. *Let R be a Ritt algebra and let I be a subset of R, then*

(1) If I is a prime ideal of R , then $I_{(n)}$ is a prime ideal of R.

(2) If I is a radical ideal of R , then $I_{(n)}$ is a radical ideal of R.

Proof. (1) From theorem 1.4 we have $I_{(n)}$ is an ideal of R , so suppose that $a \notin I_{(n)}$ and $b \notin I_{(n)}$. Then there exist positive integers m, n such that $a^{(m)} \notin I, b^{(n)} \notin I$ and for all $k < m$ and $l < n$, $a^{(k)} \in I$ and $b^{(l)} \in I$. Now let

$$(ab)^{(m+n)} = \sum_{k=0}^{m+n} \binom{m+n}{k} a^{(k)} b^{(m+n-k)}$$

We note that , $\binom{m+n}{k} a^{(k)} b^{(m+n-k)} \in I$ for $k < m$, while for $k > m$, i.e. , for $m+n-k < n$, $\binom{m+n}{k} a^{(k)} b^{(m+n-k)} \in I$.

If $k = m$, $a^{(m)} b^{(n)} \notin I$ since I is a prime ideal , and since R is a Ritt algebra $\binom{m+n}{m} a^{(m)} b^{(n)} \notin I$ .Hence $(ab)^{(m+n)} \notin I$, so that $I_{(n)}$ is a prime ideal.

(2) Note that every radical ideal of R is an intersection of prime ideals of R and conversely . Since the operator $(\ )_{(n)}$ preserves the intersection of ideals by Theorem 1.1 and prime ideals by part (1) , we have well that $(\ )_{(n)}$ preserves the radical ideals.

Definition 2.3 [7] . *Let R be a differential ring , R is said to be a Keigher ring if for each prime ideal I in R , $I_{(n)}$ is also prime ideal in R.*

Examples. **1.** Every Ritt algebra R is a Keigher ring by the above Theorem 2.2.

2. Every differential field F is a keigher ring.

3. Every ring R with trivial derivation (i.e., $a^{(n)} = 0$ for all $a \in R$ and $n \geq 1$) is a Keigher ring.

Theorem 2.4. *Let R be a Keigher differential ring and $\varphi : R \to S$ a surjective differential ring homomorphism. Then S is also a Keigher ring.*

Proof. Since φ is surjective , then φ induces a one-to-one correspondence between prime ideals J in S and prime ideals I in R containing the kernel of φ via $I = \varphi^{-1}(J)$ and $J = \varphi(I)$. Hence if J is a prime ideal in S, then we have $J_{(n)} = \varphi(\varphi^{-1}(J_{(n)})) = \varphi((\varphi^{-1}(J))_{(n)})$. But since R

is a Keigher ring , $(\varphi^{-1}(J))_{(n)}$ is prime ideal in R and hence $J_{(n)}$ is prime ideal in S.

Recall that if S is a multiplicative subset of a differential ring R , then the ring opf fractions $S^{-1}R$ is a differential ring via $(\frac{r}{s})^{(1)} = \frac{s\,r^{(1)} - r\,s^{(1)}}{s^{(2)}}$, see [2].

The following lemma was proved by Keigher in [7].

Lemma 2.5. *Let R be a differential ring . Let S be a multiplicative subset of R and I a prime ideal in R such that $I \cap S = \varnothing$. Then in the differential ring $S^{-1}R$ we have*
$(S^{-1}I)_{(n)} = S^{-1}I_{(n)}$.

Theorem 2.6. *Let R be a Keigher differential ring and S a multiplicative subset of R. Then $S^{-1}R$ is also a Keigher ring.*

Proof. The proof follows immediately from the Lemma 2.1 , since there is a one-to-one correspondence between prime ideals of $S^{-1}R$ and prime ideals of R disjoint from S [7].

Corollary 2.7. *Let R be a differential ring and let P be a prime ideal of R , then R is a Keigher differential ring if and only if R_P is a Keigher ring .*

Proof. If R is a Keigher ring , then so every R_P by Theorem 2.4. Conversely , let P be a prime ideal of R and let $f : R \to R_P$ be the canonical differential ring homomorphism . Let $S = R - P$, then since $P = f^{-1}(S^{-1}P)$, we see that $P_{(n)} = f^{-1}((S^{-1}P)_{(n)})$ by Theorem 1.1, and since R_P is a Keigher ring , $(S^{-1}P)_{(n)}$ is prime in R_P . Hence $P_{(n)}$ is prime in R and R is a Keigher ring.

Theorem 2.8. *Let $R = \prod_{i=1}^{n} R_i$, where R_i is differential ring . Then R is a Keigher ring if and only if each R_i is a Keigher ring.*

Proof. If R is a Keigher ring , then so is each R_i by Theorem 2.3 . Conversely suppose that I is a prime ideal of R , and let $\pi_i : R \to R_i$, $i = 1,2,\ldots,n$, be the canonical projections. Then $\pi_k(I) = I_k$, $1 \le k \le n$,is a prime ideal in R_k and $\pi_j(I) = R_j$ for $j \ne k$. It is clear that $I_{(n)} = \pi_k^{-1}((I_k)_{(n)})$, and since R_k is a Keigher ring , $I_{(n)}$ is prime ideal of R and R is a Keigher ring.

Definition 2.9 [5] . A differential ring R is called a $d-MP$ ring if the radical of a differential ideal I of R is again a differential ideal. This is equivalent , see [2] , [3], [8], to each of the following :

(1) Prime ideals minimal over differential ideals are differential ideals .

(2) If I is a differential ideal of R and S is a multiplicative subset of R disjoint from I , then ideals maximal among differential ideals which contain I and are disjoint from S are prime.

Theorem 2.10. *Let R be a differential ring . Then R is a Keigher ring if and only if it is a $d-MP$ ring .*

Proof. See [7] .

Let R be a differential ring . A differential ideal I is prime if and only if there is a multiplicative subset S of R such that I is maximal among ideals disjoint from S [6].

Let R be a differential ring . A differential ideal I is called quasi- prime ideal if there is a multiplicative subset S of R such that I is maximal among differential ideals disjoint from S . It is clear that every prime differential ideal is quasi-prime, and every quasi-prime ideal is prime if and only if R is a Keigher ring.

Theorem 2.11. *Let R be a differential ring . If I is a prime ideal of R then $I_{(n)}$ is a quasi-prime.*

Proof . Let I be a prime ideal of R and let $S = R - I$. It is clear that $I_{(n)}$ is a differential ideal disjoint from S and if J is any differential ideal disjoint from S , then $J \subset I$, so that $J = J_{(n)} \subset I_{(n)}$. Hence $I_{(n)}$ is maximal among differential ideals disjoint from S . Now let K be a quasi- prime ideal of R and let S be a multiplicative subset of R such that K is maximal among differential ideals disjoint from S . Then there exists a prime ideal I of R such that $K \subset I$ and $I \cap S = \varnothing$ [1] .Hence $K = K_{(u)} \subset I_{(u)}$ and $I_{(n)} \cap S = \varnothing$, so that $K = I_{(n)}$.

3. The Prime Spectrum of a differential ring

In the sense of ring theory , for any commutative ring R , Spec(R) denote the set of prime ideals in R with the Zariski topology [4]. The following two theorems show how to create a topological space from a commutative ring R . This topological space is called the prime spectrum of R and the topology is called the Zariski topology.

Theorem 3.1. *Let R be a commutative ring and let Spec(R) be the set of all prime ideals of R . For any subset A of R let $V(A)$ be the set of all prime ideals of R that contain A . Then*

(1) $V(A)=V((A))$ for any subset A of R (where (A) is the ideal generated by A).

(2) $V(0)=$ Spec (R) and $V(R)=\varnothing$.

(3) If $\{A_i\}_{i\in I}$ is a family of subsets of R , then $V(\bigcup_{i\in I} A_i)=\bigcap_{i\in I} V(A_i)$.

(4) If A and B are two subsets of R , then $V(A\cap B)=V(A)\cup V(B)$.

Parts (2),(3) and (4) show that the sets $V(A)$, as A runs over all subsets of R , satisfy the axioms for a collection of closed sets in a topological space . The subset $V(A)$ of Spec(R) are called Zarisky closed sets . Henceforth , Spec(R) is considered to have the topology defined by taking the Zariski closed sets to be the closed sets − this is the Zariski topology on Spec(R).

Theorem 3.2. *Let R and S be commutative rings and let $\varphi: R\to S$ be a ring homomorphism such that $\varphi(1)=1$ ideal of S , then $\varphi^{-1}(I)$ is a prime ideal of R . Thus φ induce a map $\varphi^*: Spec(S)\to Spec(R)$ defined by $\varphi^*(I)=\varphi^{-1}(I)$ for all $I\in Spec(S)$.*

(2) For any ideal J in R , $\varphi^{*-1}(V(J))=V((\varphi(J)))$ (where $(\varphi(J))$ is the ideal generated by $\varphi(J)$ in S). Deduce that φ^* is a continuous map with respect to the Zariski topology on Spec(S) and Spec(R).

(3) If $\Omega: S\to T$ is also a homomorphism of commutative rings , then $(\Omega\circ\varphi)^*=\varphi^*\circ\Omega^*$.

Proof. The proof follows directly from the definitions , see [4].

If R is a differential ring , the set of prime differential ideals in R will be denoted by Spec$_d(R)$ and will be called the prime differential spectrum of R . As a topologi- cal space , the set Spec$_d(R)$ has the subspace topology from Spec(R). So that the closed sets in Spec$_d(R)$ are defined by the form $V_*(A)=V(A)\cap$ Spec$_d(R)$, where A is a subset of R.

Denote by $r_d(I)$ the differential radical of differential ideal I of R and I is called a differential radical ideal if $I=r_d(I)$.

For an element $a\in R$ denote by $[a]$ the smallest differential ideal containing a .

Some of properties of differential radical ideals are given in the following theorems.

Theorem 3.3 [8] . *For a differential ring R the following conditions are equivalent :*

(1) Every differential ideal of R is differential radical ideal .

(2) $I.J=I\cap J$ for all differential ideals I,J in R .

(3) $[a]^2=[a]$ for all $a\in R$.

If $r((A))$ denotes the radical of the ideal in R generated by A , $r_d(A)$ denotes the differential radical of A , and $r_d(A)$ can be defined as following :

Theorem 3.3[8] . *For any subset A of a differential ring R , the differential radical of A , $r_d(A)$ is the intersection of all differential prime ideals in R containing A .*

It is clear that , $A\subset r((A))\subset r_d(A)$ and $r_d(r_d(A))=r_d(A)$, where A is a subset of R. If Y is a subset of Spec$_d(R)$, let $V_d(Y)$ denote the intersection of all prime differential ideals of R which belong to Y . It is easy to show that [9] :

(1) $V_d(Y)$ is a differential ideal of R, and the map from Spec$_d(R)$ to R given by $Y\mapsto V_d(Y)$ is order – reversing with respect to the partial ordering by inclusion in Spec$_d(R)$ and R.

(2) $V_d(\varnothing)=R$.

(3) If $\{Y_i\}_{i\in I}$ is a family of subsets of Spec $_d(R)$, then $V_d(\bigcup_{i\in I} Y_i) = \bigcap_{i\in I} V_d(Y_i)$.

Theorem 3.4. *Let R be a differential ring , A a subset of R , and Y a subset of $Spec_d(R)$. Then*

(1) $V_*(A)$ is closed in $Spec_d(R)$ and $V_d(Y)$ is a differential radical ideal of R.

(2) $V_d(V_*(A))$ is the differential radical of A and $V_*(V_d(Y))$ is the closure of Y in $Spec_d(R)$.

Proof. The proof follows from the definitions and the notes above.

Now let R and S be differential rings and $\psi : R \to S$ be a differential ring homomorphism . Then ψ induce a continuous map ψ^*: Spec$(S) \to$ Spec(R) given by $\psi^*(P) = \psi^{-1}(P)$ for all $P \in$ Spec(S). It follows from Theorems 1.2 , 3.2 that ψ^* restricts to give a continuous map ψ_d^* : Spec$_d(S) \to$ Spec$_d(R)$. If $\phi : S \to T$ is another differential ring homomorphism , then it is clearly that $(\phi \circ \psi)_d^* = \psi_d^* \circ \phi_d^*$.

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المثاليات ألتفاضلية للحلقات ألتفاضلية

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الخلاصة

في بحثنا هذا عرفنا مؤثرين يرمز لهما $(\ )_u$ و $(\ )_{(n)}$ لحلقة تفاضلية و قد تم بنائهما من مجموعة جزئية لتلك الحلقة . ثم ناقشنا العلاقة بين هذين المؤثرين والمثاليات التفاضلية لحلقة تفاضلية وبشكل خاصحلقة كيكر التفاضلية .