P- ISSN 1991-8941 E-ISSN 2706-6703 *Journal of University of Anbar for Pure Science (JUAPS)* Open Access
2022,(16), (2):99– 107

Double Intuitionistic Open Sets with Some Applications

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ARTICLE INFO
Received: 23 / 10 / 2022
Accepted: 21 / 11 / 2022
Available online: 23 /12/ 2022
DOI:10.37652/juaps.2022.176496

Keywords:
DIS, DITS, Double I point, Double I interior set, Double I closure set and, Double I limit point in DITS.

ABSTRACT:

The purpose of this work is to presents a new class of open sets namely Double intuitionistic open sets. The relationships between the Double intuitionistic open and the Double intuitionistic sets are studied including the Double intuitionistic interior set, Double intuitionistic closure set and Double intuitionistic limit point in Double intuitionistic topological spaces were presented and various examples and many observations were presented for each concept, also the definitions of the paired set were presented in general topological spaces Therefore, we generalize it to the Double intuitionistic topological spaces with the presentation of the basic theorem of this new space, and many of the basic characteristics and properties related to these concepts that are presented in the third section as evidence with many examples, taking into account that the opposite is not true for each evidence for these characteristics, which was presented and what we note in the fourth section.

1. INTRODUCTION

The concept of general topological spaces, their types, and basic concepts were introduced by step by step [1]. Abbas and El-Sanousy [2] introduced and studied several types of Double fuzzy semi closed sets. Al-Yasry, A.Z. [3] investigated the properties of lectures in advanced topology. The theory of intuitionistic fuzzy sets was examined and developed more on fuzzy sets [4]. Later, the concept was used to define intuitionistic sets and on intuitionistic gradation of openness by Coker [5,6]. El-Maghrabi, A. I. T., & Al-Juhani, M.A. N. [7] extended the notion of Some applications of M-open sets in topological spaces. Ghareeb, A. [8] introduce and characterize several types of normality in double fuzzy topological spaces and the effects of some types of functions on these types of normality are introduced. Generalization of the concept of double set was first introduced by Kandil, Tantawy, and Wafaie on flou intuitionistic topological spaces [9]. In Mohammed, F. M., Abdullah, S. I., & Obaid, S. H. [10] investigated (p, q)-fuzzy αm-closed sets in double fuzzy topological spaces. The concept of on intuitionistic gradation of openness.

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Fuzzy sets and systems which was presented by Mondal, T.K., & Samanta, S. K [11]. Ozcelik and Narli [12] introduced and investigated the concept on submaximality intuitionistic topological spaces.

Also introduced the concept of Double sets and Double continuous function in Double intuitionistic topological spaces and investigated basic properties of generalization Double intuitionistic set by Raoof, A. G., & Jassim, T. H [13]. In [14] presented some types of fuzzy Z closed sets in double fuzzy topological spaces. Viro, O. Y., Ivanov, O. A., Netsvetaev, N. Y., & Kharlamov, V. M. [15] studied and submitted the concept of elementary topology problem textbook.

The purpose of this paper is to introduce a new class of sets in (DITS) namely Double I-open sets with some applications in DITS which is between the class of Double I-int Ψ set, Double I-clΨ set and the class Double I-limit point Ψ set (see section 3). In section 4, we study the basic characteristics and qualities related to these types and the relationships between them and give examples the converse is not true.

2- Preliminaries

We recall the following definitions, which are needed, in our work.

Definition 2.1 [5] Let $X \neq \emptyset$, and let $\mathfrak{P}$ and $\mathfrak{Q}$ be IS having the form $\mathfrak{P} = \langle x, \mathfrak{P}_1, \mathfrak{P}_2\rangle, \mathfrak{Q} = \langle x, \mathfrak{Q}_1, \mathfrak{Q}_2\rangle$ respectively. Also, $\{\mathfrak{P}_i : i \in I\}$ be an arbitrary family of IS in X, where $\mathfrak{P}_i = \langle x, \mathfrak{P}_i^{(1)}, \mathfrak{P}_i^{(2)}\rangle$, afterward:

1) $\widetilde{\emptyset} = \langle x, \emptyset, X\rangle$; $\widetilde{X} = \langle x, X, \emptyset\rangle$.

2) $\mathfrak{P} \subseteq \mathfrak{Q}$ iff $\mathfrak{P}_1 \subseteq \mathfrak{Q}_1$ and $\mathfrak{Q}_2 \supseteq \mathfrak{P}_2$.

3) $\mathfrak{P}^{c} = \langle x, \mathfrak{P}_2, \mathfrak{P}_1\rangle$.

4) $\cup \mathfrak{P}_i = \langle x, \cup \mathfrak{P}_i^{(1)}, \cap \mathfrak{P}_i^{(2)}\rangle$, $\cap \mathfrak{P}_i = \langle x, \cap \mathfrak{P}_i^{(1)}, \cup \mathfrak{P}_i^{(2)}\rangle$.

5) $\mathfrak{P} - \mathfrak{Q} = \mathfrak{P} \cap \mathfrak{Q}^{c}$.

6) $\mathfrak{P} = \mathfrak{Q}$ if and only if $\mathfrak{P} \subseteq \mathfrak{Q}$ and $\mathfrak{Q} \subseteq \mathfrak{P}$.

Definition 2.2 [12] Let X be a non-empty set, an intuitionistic set $\mathfrak{P}$ (IS, for short) is an object having g the form $\mathfrak{P} = \langle x, \mathfrak{P}_1, \mathfrak{P}_2\rangle$ where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are disjoint subset of X. Then $\mathfrak{P}_1$ is called set of members of $\mathfrak{P}$, while $\mathfrak{P}_2$ is called set of nonmembers of $\mathfrak{P}$ [4]. An intuitionistic topology (IT, for short) on a non-empty set X, is a family T of IS in X containing $\widetilde{\emptyset}, \widetilde{X}$ and closed under arbitrary unions and finitely intersections. The pair (X, T) is called ITS.

Definition 2.3 [9] Let $X \neq \emptyset$.

1) A Double-set (D- set, for short) $\underline{\mathfrak{U}}$ is an ordered pair $(\mathfrak{U}_1, \mathfrak{U}_2) \in \wp(X) \times \wp(X)$ such that $\mathfrak{U}_1 \subseteq \mathfrak{U}_2$.

2) D (X) = $\{(\mathfrak{U}_1, \mathfrak{U}_2) \in \wp(X) \times \wp(X), \mathfrak{U}_1 \subseteq \mathfrak{U}_2\}$ is the family of all D-sets on X .

3) The D-set $\underline{X}$ = (X,X) is called the universal D-set, and the D-set $\underline{\emptyset} = (\emptyset,\emptyset)$ is called the empty D-set.

4) Let $\eta_1, \eta_2 \subseteq \wp(X)$. The product of η_1 and η_2, denoted by $\eta_1 \times \eta_2$ defined by $\eta_1 \times \eta_2 = \{(\mathfrak{U}1, \mathfrak{U}2) : \mathfrak{U}_1 \in \eta_1, \mathfrak{U}_2 \in \eta_2, \mathfrak{U}_1 \subseteq \mathfrak{U}_2\}$.

5) Let $\mathfrak{U} = (\mathfrak{U}_1, \mathfrak{U}_2)$; $\underline{\vartheta} = (\vartheta_1, \vartheta_2) \in D(X)$:

1) $(\underline{\mathfrak{U}}^{c}) = (\mathfrak{U}^{c}_2, \mathfrak{U}^{c}_1)$ where $\underline{\mathfrak{U}}^{c}$ is the complement of $\underline{\mathfrak{U}}$.

2) $\underline{\mathfrak{U}} - \underline{\vartheta} = (\mathfrak{U}_1 - \vartheta_2, \mathfrak{U}_2 - \vartheta_1)$.

Definition 2.4 [9] Let X be a non-empty set. The family η of D-sets in X is called a double topology on X if it satisfies the following axioms: a) $\underline{\emptyset}$, $\underline{X} \in \eta$.

b) If $\underline{\mathfrak{U}}$, $\underline{\vartheta} \in \eta$, then $\underline{\mathfrak{U}} \cap \underline{\vartheta} \in \eta$,

c) If $\{\underline{\mathfrak{U}z} : z \in Z\} \subseteq \eta$, then $\cup_{z \in Z} \underline{\mathfrak{U}z} \in \eta$. The pair (X, η) is called a DTS. Each element of η is called an open D-set in X . The complement of open D-set is called closed D-set.

Definition 2.5 [9] Let X be a non-empty set defined by:

1) IN (X) = $\{\underline{\emptyset}, \underline{X}\}$, then IN is a Double topology on X and is called indiscrete Double topology. (X, IN) is called indiscrete Double space.

2) dis (X) = $\wp(X) \times \wp(X)$ (power set of X's, then dis is a Double topology on X and is called discrete Double topology. (X, dis) is called discrete Double space.

Definition 2.6 [1], [8] Let (X, η) be a DTS and $\underline{\mathfrak{U}} \in D(X)$. The double closure and interior of $\mathfrak{U}$, denoted by cl $(\underline{\mathfrak{U}})$, int $(\underline{\mathfrak{U}})$ defined by: cl $(\underline{\mathfrak{U}}) = \cap \{\underline{\vartheta} : \underline{\vartheta} \in \eta^{c}$ and $\mathfrak{U} \subseteq \vartheta\}$, int $(\underline{\mathfrak{U}}) = \cup \{\underline{G_i} : \underline{G_i} \in \eta$ and $G \subseteq \mathfrak{U}\}$.

Definition 2.7 [1], [2],[15] Let (X, η) be a DTS and $\underline{\mathfrak{U}} \in$ D (X). A point $x \in X$ is a limit point or cluster point of $\mathfrak{U}$ and is denoted by $\mathfrak{U}'$ is the set $\mathfrak{U}' = \{x \in \underline{X} : \forall \vartheta \in \tau; x \in \vartheta \wedge \vartheta \setminus \{x\} \cap \mathfrak{U} \neq \phi\}$.

Definition 2.8 [6] Let X nonempty set, $p \in X$ a fixed element in X, and let $\mathfrak{P} = \langle x, \mathfrak{P}_1, \mathfrak{P}_2\rangle$ be an intuitionistic set (IS, for short). The IS $\dot{p}$ defined by $\dot{p} = \langle x, \{p\}, \{p\}^{c}\rangle$ is called an intuitionistic point (Ip, for short) in X. The IS $\ddot{p}$ $= \langle x, \emptyset, \{p\}^{c}\rangle$ is called a vanishing intuitionistic point (VIp, for short) in X. The IS $\dot{p}$ is said to be contained in $\mathfrak{P}$ ($\dot{p} \in \mathfrak{P}$, for short) if and only if $p \in \mathfrak{P}_1$, and similarly IS $\ddot{p}$ contained in $\mathfrak{P}$. ($\ddot{p} \in \mathfrak{P}$, for short) if and only if $p \notin \mathfrak{P}_2$. For a given IS p in X, we may write $\mathfrak{P} = (\cup\{p : \dot{p} \in \mathfrak{P}\}) \cup (\cup\{\ddot{p} : \ddot{p} \in \mathfrak{P}\})$, and whenever $\mathfrak{P}$ is not a proper IS (i.e., if $\mathfrak{P}$ is not of the form $\mathfrak{P} = \langle x, \mathfrak{P}_1, \mathfrak{P}_2\rangle$ where $\mathfrak{P}_1 \cup \mathfrak{P}_2 \neq X$), then $\mathfrak{P} = \cup\{\dot{p} : \dot{p} \in \mathfrak{P}\}$ hold . In general, any IS $\mathfrak{P}$ in X can be written in the form $\mathfrak{P} = \dot{\mathfrak{P}} \cup \ddot{\mathfrak{P}}$ where $\dot{\mathfrak{P}} = \cup\{\dot{\mathfrak{P}} : \dot{\mathfrak{P}} \in \mathfrak{P}\}$ and $\ddot{\mathfrak{P}} = \cup\{\ddot{p} : \ddot{p} \in \mathfrak{P}\}$.

Definition 2.9 [13] Let $\mathbb{X}$ be a non-empty set.

1) A Double intuitionistic set (Double I-set, for short) is an ordered pair $(\mathcal{Q}, \mathcal{D}) = (\langle x, \mathcal{Q}_1, \mathcal{Q}_2\rangle, \langle x, \mathcal{D}_1, \mathcal{D}_2\rangle) \in$ $pI(\mathbb{X}) \times pI(\mathbb{X})$ such that $\mathcal{Q} \subseteq \mathcal{D}$.

2) Double I $(\mathbb{X}) = \{(\mathcal{Q}, \boldsymbol{\mathcal{D}}) \in pI(\mathbb{X}) \times pI(\mathbb{X}), \mathcal{Q} \subseteq \mathcal{D}\}$ is the family of all Double I-sets on $\mathbb{X}$.

3) The Double I-set $(\langle x, \mathbb{X}, \emptyset\rangle, \langle x, \mathbb{X}, \emptyset\rangle) = (\widetilde{\mathbb{X}}, \widetilde{\mathbb{X}})$ is called the universal Double I-set, and the Double I-set $(\widetilde{\emptyset}, \widetilde{\emptyset}) =$ $(\langle x, \emptyset, \mathbb{X}\rangle, \langle x, \emptyset, \mathbb{X}\rangle)$ is called the empty Double I-set.

4) Let $\Psi_1, \Psi_2 \subseteq pI(\mathbb{X})$. The Double product of Ψ_1 and Ψ_2,

defined by $\Psi_1 \times \Psi_2 = \{(Q,\mathcal{D}) : \mathcal{Q} \in \Psi_1, \mathcal{D} \in \Psi_2, Q \subseteq \mathcal{D}\}$.

5 5) Let $(\mathcal{Q},\mathcal{D}), (\mathcal{C},\mathcal{G}) \in$ Double I (Ӿ):

1) $(\mathcal{Q},\mathcal{D})^c = (\mathcal{D}^c, \mathcal{Q}^c) = (\langle x, \mathcal{D}_1, \mathcal{D}_2\rangle^c, \langle x, \mathcal{Q}_1, \mathcal{Q}_2\rangle^c) = (\langle x, \mathcal{D}_2, \mathcal{D}_1\rangle, \langle x, \mathcal{Q}_2, \mathcal{Q}_1\rangle)$.
2) $(\mathcal{Q},\mathcal{D}) \setminus (\mathcal{C},\mathcal{G}) = ((\mathcal{Q} \setminus \mathcal{G}), (\mathcal{D} \setminus \mathcal{C})) = (\langle x, \mathcal{Q}_1, \mathcal{Q}_2\rangle, \langle x, \mathcal{D}_1, \mathcal{D}_2\rangle) \setminus (\langle x, \mathcal{C}_1, \mathcal{C}_2\rangle, \langle x, \mathcal{G}_1, \mathcal{G}_2\rangle) = ((\langle x, \mathcal{Q}_1, \mathcal{Q}_2\rangle \setminus \langle x, \mathcal{G}_1, \mathcal{G}_2\rangle), (\langle x, \mathcal{D}_1, \mathcal{D}_2\rangle, \langle x, \mathcal{C}_1, \mathcal{C}_2\rangle)$. Each element of Ψ is called a Double intuitionistic open set (DIOS, for short) in Ӿ. The complement of DIOS is called Double intuitionistic closed set (DICS, for short).

Theorem 2.10 [13] Let Ӿ $\neq \emptyset$, then the family T of all Double intuitionistic open sets in Ӿ is Double intuitionistic topological spaces (DITS)**.**

Proof Let (Ӿ, T) be intuitionistic topological spaces (ITS), then:

1) $\widetilde{\emptyset} = \langle x, \emptyset, Ӿ\rangle$, $\widetilde{Ӿ} = \langle x, Ӿ, \emptyset\rangle \in$ IT $\rightarrow$ $(\widetilde{\emptyset},\widetilde{\emptyset})$, $(\widetilde{Ӿ},\widetilde{Ӿ}) \in$ DITS.

2) Let $(\mathcal{Q},\mathcal{D}), (\mathcal{C},\mathcal{G}) \in$ DIT $\rightarrow \mathcal{Q}, \mathcal{D}, \mathcal{C}, \mathcal{G} \in$ IT. Since IT is intuitionistic topology, then $\mathcal{Q} \cap \mathcal{D} \in$ IT and $\mathcal{C} \cap \mathcal{G} \in$ IT. Now, let $\mathcal{K} = (\mathcal{Q},\mathcal{D})$ and $\mathcal{W} = (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{K},\mathcal{W}) = ((\mathcal{Q},\mathcal{C}), (\mathcal{D},\mathcal{G})) \in$ DITS.

3) Let $(\mathcal{Q}_s, \mathcal{D}_s)$ be a family of IS and s ∈ S and $(\mathcal{Q}_s, \mathcal{D}_s) \in$ DIT $\rightarrow$ $\mathcal{Q}_s, \mathcal{D}_s \in$ ITS, since IT is intuitionistic topology, then $\cup_{s\in S}\, \mathcal{Q}_s \in$ IT and $\cup_{s\in S}\, \mathcal{D}_s \in$ IT. Thus $\cup_{s\in S}\, (\mathcal{Q}_s, \mathcal{D}_s) \in$ DIT. Therefore, (Ӿ, T) is Double intuitionistic topological spaces.

3- Double intuitionistic open sets in DITS

In this section, we introduce a new class of DIS in Double intuitionistic topological spaces and related to other kind of sets, which are defined in this work. We begin this section by the following definitions.

Definition 3.1 Let Ӿ nonempty set, $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in$ Ӿ a fixed element in Ӿ, and let $(\mathcal{Q},\mathcal{D}) = (\langle x, \mathcal{Q}_1, \mathcal{Q}_2\rangle, \langle x, \mathcal{D}_1, \mathcal{D}_2\rangle)$ be a Double intuitionistic set (Double I-set). The Double I-set $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})$ defined by $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) = (\langle x, \{\mathfrak{p}\}, \{\mathfrak{p}\}^c\rangle, \langle x, \{\mathfrak{p}\}, \{\mathfrak{p}\}^c\rangle)$ is called a Double intuitionistic point $(Double\ I-\mathfrak{p}oint, for\ short)$ in Ӿ. The Double I-set $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})$ *is* said to be contained in $(\mathcal{Q},\mathcal{D})$ if and only if $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q}_1,\mathcal{D}_1)$.

Definition 3.2 Let (Ӿ,Ψ) be a DITS and $(\mathcal{Q},\mathcal{D}) \in$ Double I(Ӿ), then the Double interior of $(\mathcal{Q},\mathcal{D})$ is the Double I-set such that int $(\mathcal{Q},\mathcal{D}) = \cup \{(\mathcal{C},\mathcal{G}) : (\mathcal{C},\mathcal{G}) \in \Psi$ and $(\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D})\}$.

Remarks 3.3 1) As int $(\mathcal{Q},\mathcal{D})$ is the union of all Double intuitionistic open sets contained in $(\mathcal{Q},\mathcal{D})$. it is the largest DIOS contained in $(\mathcal{Q},\mathcal{D})$. So int $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D})$.

2) If $(\mathcal{C},\mathcal{G})$ is a DIOS such that $(\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D})$ then $(\mathcal{C},\mathcal{G}) \subseteq$ int $(\mathcal{Q},\mathcal{D})$.

3) The Double interior of $(\mathcal{Q},\mathcal{D})$ will be denoted by $(\mathcal{Q},\mathcal{D})^0$.

4) $(\mathcal{Q},\mathcal{D})$ is a DIOS iff each Double point $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})$ is a Double interior point of $(\mathcal{Q},\mathcal{D})$ iff there exist a DIOS $(\upsilon,\nu) \in \Psi$ containing $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})$ such that $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\upsilon,\nu) \subseteq (\mathcal{Q},\mathcal{D})$.

Definition 3.4 Let (Ӿ, Ψ) be a DITS and $(\mathcal{Q},\mathcal{D}) \in$ Double I(Ӿ), the Double closure of $(\mathcal{Q},\mathcal{D})$ denoted by cl $(\mathcal{Q},\mathcal{D})$ or $\overline{(\mathcal{Q},\mathcal{D})}$ is the Double I-set such that cl $(\mathcal{Q},\mathcal{D}) = \cap \{(\mathcal{C},\mathcal{G}) : (\mathcal{C},\mathcal{G}) \in \Psi^c$ and $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G})\}$.

Remarks 3.5 1) In a DITS (Ӿ,IN) if $(\widetilde{\emptyset},\widetilde{\emptyset}) \neq (\mathcal{Q},\mathcal{D}) \subseteq (\widetilde{Ӿ},\widetilde{Ӿ})$, then cl $(\mathcal{Q},\mathcal{D}) = (\widetilde{Ӿ},\widetilde{Ӿ})$. Since cl $(\mathcal{Q},\mathcal{D})$ is Double I-closed set in IN contain $(\mathcal{Q},\mathcal{D}) = (\widetilde{Ӿ},\widetilde{Ӿ})$.

2) In a DITS (Ӿ,dis), every subset of $(\widetilde{Ӿ},\widetilde{Ӿ})$ is Double I-open and Double I-closed in the sometime, then cl $(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D})$ for all $(\mathcal{Q},\mathcal{D}) \subseteq (\widetilde{Ӿ},\widetilde{Ӿ})$.

Definition 3.6 Let (Ӿ, Ψ) be a DITS and $(\mathcal{Q},\mathcal{D}) \in$ Double I (Ӿ). A Double intuitionistic point $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in$ Ӿ is called Double limit point of $(\mathcal{Q},\mathcal{D})$ iff every Double I-open set containing $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})$ contains at least one Double I-point of $(\mathcal{Q},\mathcal{D})$ different from $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})$. The Double I-set of all Double limit point of $(\mathcal{Q},\mathcal{D})$ is called Double derived set of $(\mathcal{Q},\mathcal{D})$ and is denoted by $(\mathcal{Q},\mathcal{D})'$ i.e., $(\mathcal{Q},\mathcal{D})' = \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in Ӿ : \forall (\upsilon,\nu) \in \Psi ; \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\upsilon,\nu) \wedge (\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap (\mathcal{Q},\mathcal{D}) \neq (\widetilde{\emptyset},\widetilde{\emptyset})$. If $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \notin (\mathcal{Q},\mathcal{D})' \Leftrightarrow \exists (\upsilon,\nu) \in \Psi ; \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\upsilon,\nu) \wedge (\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap (\mathcal{Q},\mathcal{D}) = (\widetilde{\emptyset},\widetilde{\emptyset})$.

Remarks 3.7 Let Ӿ $\neq (\widetilde{\emptyset},\widetilde{\emptyset})$ and $\Psi = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{Ӿ},\widetilde{Ӿ})\}$. This

P- ISSN 1991-8941 E-ISSN 2706-6703 Journal of University of Anbar for Pure Science (JUAPS) Open Access
2022,(16), (2):99– 107

Ψ is topology on Ӿ called Double indiscrete topology.

1) choose $(\mathcal{Q},\mathcal{D}) = (\widetilde{\emptyset},\widetilde{\emptyset}) \subseteq$ Ӿ $\rightarrow (\widetilde{\emptyset},\widetilde{\emptyset})' = (\widetilde{\emptyset},\widetilde{\emptyset})$. So $(\widetilde{\emptyset},\widetilde{\emptyset})$ does not Double limit point, since for every Double I-open set (υ,ν) any for every element $\{(\tilde{p},\tilde{p})$ is $(\upsilon,\nu) \setminus \{(\tilde{p},\tilde{p})\} \cap (\widetilde{\emptyset},\widetilde{\emptyset}) = (\widetilde{\emptyset},\widetilde{\emptyset})$.

2) choose $(\mathcal{Q},\mathcal{D}) \neq (\widetilde{\emptyset},\widetilde{\emptyset})$ and $(\mathcal{Q},\mathcal{D})$ contains more than one element, then $(\mathcal{Q},\mathcal{D})' = (\widetilde{Ӿ},\widetilde{Ӿ})$, because the only Double I-open set indiscrete is Ӿ for every element in Ӿ and $(\widetilde{Ӿ},\widetilde{Ӿ}) \setminus \{(\tilde{p},\tilde{p})\} \cap (\mathcal{Q},\mathcal{D}) \neq (\widetilde{\emptyset},\widetilde{\emptyset})$.

In the next example we show that Double interior (Double closure and Double limit point) in DITS.

Example 3.8 Let Ӿ $= \{i, j, h\}$; $\Psi = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{Ӿ},\widetilde{Ӿ}), (\mathcal{M}_1,\mathcal{M}_2), (\mathcal{M}_3,\mathcal{M}_4), (\mathcal{M}_1,\mathcal{M}_4), (\mathcal{M}_4,\widetilde{Ӿ}), (\widetilde{\emptyset},\mathcal{M}_1), (\mathcal{M}_1,\widetilde{Ӿ}), (\mathcal{M}_1,\mathcal{M}_1), (\widetilde{\emptyset},\mathcal{M}_4), (\mathcal{M}_4,\mathcal{M}_4)\}$ where $(\mathcal{M}_1,\mathcal{M}_2) =$

$(\langle x,\{j\},\{i,h\}\rangle, \langle x,\{i,j\},\{h\}\rangle)$, $(\mathcal{M}_3,\mathcal{M}_4) = (\langle x,\{h\},\{i,j\}\rangle, \langle x,\{j,h\},\{i\}\rangle)$, $(\mathcal{M}_1,\mathcal{M}_4) = (\langle x,\{j\},\{i,h\}\rangle, \langle x,\{j,h\},\{i\}\rangle)$, $(\mathcal{M}_4,\widetilde{Ӿ}) = (\langle x,\{j,h\},\{i\}\rangle, \langle x, Ӿ, \emptyset\rangle)$, $(\widetilde{\emptyset},\mathcal{M}_1) = (\langle x,\emptyset, Ӿ\rangle, \langle x,\{j\},\{i,h\}\rangle)$, $(\mathcal{M}_1,\widetilde{Ӿ}) = (\langle x,\{j\},\{i,h\}\rangle, \langle x, Ӿ,\emptyset\rangle)$, $(\mathcal{M}_1,\mathcal{M}_1) = (\langle x,\{j\},\{i,h\}\rangle, \langle x,\{j\},\{i,h\}\rangle)$, $(\widetilde{\emptyset},\mathcal{M}_4) = (\langle x,\emptyset, Ӿ\rangle, \langle x,\{j,h\},\{i\}\rangle)$ and $(\mathcal{M}_4,\mathcal{M}_4) = (\langle x,\{j,h\},\{i\}\rangle, \langle x,\{j,h\},\{i\}\rangle)$. $\Psi^c = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{Ӿ},\widetilde{Ӿ}), (\mathcal{M}_2^c,\mathcal{M}_1^c), (\mathcal{M}_4^c,\mathcal{M}_3^c), (\mathcal{M}_4^c,\mathcal{M}_1^c), (\widetilde{\emptyset},\mathcal{M}_4^c), (\mathcal{M}_1^c,\widetilde{Ӿ}), (\widetilde{\emptyset},\mathcal{M}_1^c), (\mathcal{M}_1^c,\mathcal{M}_1^c), (\mathcal{M}_4^c,\widetilde{Ӿ}), (\mathcal{M}_4^c,\mathcal{M}_4^c)\}$ where $(\mathcal{M}_2^c,\mathcal{M}_1^c) = (\langle x,\{h\},\{i,j\}\rangle, \langle x,\{i,h\},\{j\}\rangle)$, $(\mathcal{M}_4^c,\mathcal{M}_3^c) = (\langle x,\{i\},\{j,h\}\rangle, \langle x,\{i,j\},\{h\}\rangle)$, $(\mathcal{M}_4^c,\mathcal{M}_1^c) = (\langle x,\{i\},\{j,h\}\rangle, \langle x,\{i,h\},\{j\}\rangle)$, $(\widetilde{\emptyset},\mathcal{M}_4^c) = (\langle x,\emptyset, Ӿ\rangle, \langle x,\{i\},\{j,k\}\rangle)$, $(\mathcal{M}_1^c,\widetilde{Ӿ}) = (\langle x,\{i,h\},\{j\}\rangle, \langle x, Ӿ,\emptyset\rangle)$, $(\widetilde{\emptyset},\mathcal{M}_1^c) = (\langle x,\emptyset, Ӿ\rangle, \langle x,\{i,h\},\{j\}\rangle)$, $(\mathcal{M}_1^c,\mathcal{M}_1^c) = (\langle x,\{i,h\},\{j\}\rangle, \langle x,\{i,h\},\{j\}\rangle)$, $(\mathcal{M}_4^c,\widetilde{Ӿ}) = (\langle x,\{j,h\},\{i\}\rangle, \langle x, Ӿ,\emptyset\rangle)$ and $(\mathcal{M}_4^c,\mathcal{M}_4^c) = (\langle x,\{i\},\{j,h\}\rangle, \langle x,\{i\},\{j,h\}\rangle)$. Let $(\mathcal{M}_4^c,\mathcal{M}_2) = (\langle x,\{i\},\{j,h\}\rangle, \langle x,\{i,j\},\{h\}\rangle)$ is Double interior set in Ӿ, since the union of all Double I-open set contained in $(\mathcal{M}_4^c,\mathcal{M}_2)$ and $\mathrm{int}\,(\mathcal{M}_4^c,\mathcal{M}_2) = (\widetilde{\emptyset},\mathcal{M}_1)$ and $cl(\mathcal{M}_4^c,\mathcal{M}_2) = \cap\ \{(\mathcal{M}_4^c,\mathcal{M}_3^c), (\mathcal{M}_4^c,\mathcal{M}_3^c) \in \Psi^c$ and $(\mathcal{M}_4^c,\mathcal{M}_2) \subseteq (\mathcal{M}_4^c,\mathcal{M}_3^c)\}$. Therefore $cl(\mathcal{M}_4^c,\mathcal{M}_2) = (\mathcal{M}_4^c,\mathcal{M}_3^c)$. Let $(\mathcal{M}_1,\mathcal{M}_1)$ to find the Double limit point of any Double I-set must choose every Double I-open sets for every Double I-point in Ӿ and notes satisfy the definition or not.

$(\tilde{i},\tilde{i}) = (\langle x,\{i\},\{j,h\}\rangle, \langle x,\{i\},\{j,h\}\rangle) \in$ Ӿ and the Double I-set containing $(\tilde{i},\tilde{i})$ is $(\widetilde{Ӿ},\widetilde{Ӿ})$ only.

$(\tilde{j},\tilde{j}) = (\langle x,\{j\},\{i,h\}\rangle, \langle x,\{j\},\{i,h\}\rangle) \in$ Ӿ and the Double I-set containing $(\tilde{j},\tilde{j})$ are $(\widetilde{Ӿ},\widetilde{Ӿ})$, $(\mathcal{M}_1,\mathcal{M}_2)$, $(\mathcal{M}_1,\mathcal{M}_4)$, $(\mathcal{M}_4,\widetilde{Ӿ})$, $(\mathcal{M}_1,\widetilde{Ӿ})$, $(\mathcal{M}_1,\mathcal{M}_1)$ and $(\mathcal{M}_4,\mathcal{M}_4)$.

$(\tilde{h},\tilde{h}) = (\langle x,\{h\},\{i,j\}\rangle, \langle x,\{h\},\{i,j\}\rangle) \in$ Ӿ and the Double I-set containing $(\tilde{h},\tilde{h})$ are $(\widetilde{Ӿ},\widetilde{Ӿ})$, $(\mathcal{M}_3,\mathcal{M}_4)$, $(\mathcal{M}_4,\widetilde{Ӿ})$ and $(\mathcal{M}_4,\mathcal{M}_4)$. Notes that $(\widetilde{Ӿ},\widetilde{Ӿ}) \setminus \{((\tilde{i},\tilde{i})\} \cap (\mathcal{M}_1.\mathcal{M}_1) = (\mathcal{M}_1.\mathcal{M}_1) \rightarrow (\tilde{i},\tilde{i}) \in (\mathcal{M}_1.\mathcal{M}_1)'$.

Notes that $(\mathcal{M}_1,\mathcal{M}_2) \setminus \{((\tilde{j},\tilde{j})\} \cap (\mathcal{M}_1,\mathcal{M}_1) = (\widetilde{\emptyset},\widetilde{\emptyset}) \rightarrow (\tilde{j},\tilde{j}) \notin (\mathcal{M}_1,\mathcal{M}_1)'$. Notes that $(\widetilde{Ӿ},\widetilde{Ӿ}) \setminus \{(\tilde{h},\tilde{h})\} \cap (\mathcal{M}_1,\mathcal{M}_1) = (\mathcal{M}_1,\mathcal{M}_1)$. Notes that $(\mathcal{M}_3,) \setminus \{(\tilde{h},\tilde{h})\} \cap (\mathcal{M}_1,\mathcal{M}_1) = (\widetilde{\emptyset},\mathcal{M}_1)$. Notes that $(\mathcal{M}_4,\widetilde{Ӿ}) \setminus \{(\tilde{h},\tilde{h})\} \cap (\mathcal{M}_1,\mathcal{M}_1) = (\mathcal{M}_1,\mathcal{M}_2)$.

Notes that $(\mathcal{M}_4,\mathcal{M}_4) \setminus \{(\tilde{h},\tilde{h})\} \cap (\mathcal{M}_1,\mathcal{M}_1) = (\mathcal{M}_1,\mathcal{M}_1) \rightarrow (\tilde{h},\tilde{h}) \in (\mathcal{M}_1,\mathcal{M}_1)'$. Therefore $(\mathcal{M}_1,\mathcal{M}_1)' = \{(\tilde{i},\tilde{i}), (\tilde{h},\tilde{h})\}$.

4 -Characterizations of Topology by (Double Interior, Double Closure and Double Limit point) Operations in DITS.

Finally, we will present the basic qualities and characteristics related to the definitions that were presented in the third section as proofs and give examples of the contrary being not true for these characteristics.

Theorem 4.1 Let (Ӿ,Ψ) be a DITS and $((\mathcal{Q},\mathcal{D}), (\mathcal{C},\mathcal{G})) \in$ Double I (Ӿ). The following characterizations are hold:

1) $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{Q},\mathcal{D})$.

2) $(\mathcal{Q},\mathcal{D})^{\circ} = \cup \{(\mathcal{C},\mathcal{G}) \in \Psi; (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D})\}$. This means $(\mathcal{Q},\mathcal{D})^{\circ}$ is the large Double I-open set contain in $(\mathcal{Q},\mathcal{D})$.

3) $(\mathcal{Q},\mathcal{D})$ is Double I-open $\Leftrightarrow (\mathcal{Q},\mathcal{D})^{\circ} = (\mathcal{Q},\mathcal{D})$.

4) $(\widetilde{\emptyset},\widetilde{\emptyset})^{\circ} = (\widetilde{\emptyset},\widetilde{\emptyset})$; $(\widetilde{Ӿ},\widetilde{Ӿ})^{\circ} = (\widetilde{Ӿ},\widetilde{Ӿ})$; $((\mathcal{Q},\mathcal{D})^{\circ})^{\circ} = (\mathcal{Q},\mathcal{D})^{\circ}$.

5) If $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G})$, then $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{C},\mathcal{G})^{\circ}$.

6) $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ} = (\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ}$.

7) $(\mathcal{Q},\mathcal{D})^{\circ} \cup (\mathcal{C},\mathcal{G})^{\circ} \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))^{\circ}$.

Proof 1) $(\mathcal{Q},\mathcal{D})^{\circ} = \cup_{i=1}^{n} (\mathcal{Q}_i,\mathcal{D}_i)$; $(\mathcal{Q},\mathcal{D})$ is Double I-open subset of $(\mathcal{Q},\mathcal{D})$, so $(\mathcal{Q}_1,\mathcal{D}_1) \subseteq (\mathcal{Q},\mathcal{D})$ $(\mathcal{Q}_2,\mathcal{D}_2) \subseteq (\mathcal{Q},\mathcal{D}) \ldots (\mathcal{Q}_n,\mathcal{D}_n) \subseteq (\mathcal{Q},\mathcal{D}) \rightarrow$ $(\mathcal{Q}_1,\mathcal{D}_1) \cup (\mathcal{Q}_2,\mathcal{D}_2) \cup \ldots \cup (\mathcal{Q}_n,\mathcal{D}_n) \subseteq (\mathcal{Q},\mathcal{D})$.Since $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{Q},\mathcal{D}) \rightarrow$ $(\mathcal{Q}_1,\mathcal{D}_1) \cup (\mathcal{Q}_2,\mathcal{D}_2) \cup \ldots \cup (\mathcal{Q}_n,\mathcal{D}_n) = (\mathcal{Q},\mathcal{D})^{\circ} \rightarrow$ $(\mathcal{Q}_1,\mathcal{D}_1) \subseteq (\mathcal{Q},\mathcal{D})^{\circ}, (\mathcal{Q}_2,\mathcal{D}_2) \subseteq (\mathcal{Q},\mathcal{D})^{\circ} \ldots (\mathcal{Q}_n,\mathcal{D}_n) \subseteq (\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{Q},\mathcal{D})$. Hence $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{Q},\mathcal{D})$.

2) Since $(\mathcal{Q},\mathcal{D})^{\circ}$ is the union of all Double I-open sets contained in $(\mathcal{Q},\mathcal{D})^{\circ} \rightarrow (\mathcal{Q},\mathcal{D})^{\circ}$ is the largest Double I- open set contained in $(\mathcal{Q},\mathcal{D})$.

3) $(\Rightarrow)$ Suppose that $(\mathcal{Q},\mathcal{D})$ is Double I-open, to prove $(\mathcal{Q},\mathcal{D})^{\circ} = (\mathcal{Q},\mathcal{D})$. Since $(\mathcal{Q},\mathcal{D})$ is Double I-open, $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D})$ (i.e., $(\mathcal{Q},\mathcal{D})$ is a Double I-open set contained in $(\mathcal{Q},\mathcal{D}) \rightarrow (\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D})^{\circ}$. And $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{Q},\mathcal{D})$ (by definition of $(\mathcal{Q},\mathcal{D})^{\circ}$). Thus $(\mathcal{Q},\mathcal{D})^{\circ} = (\mathcal{Q},\mathcal{D})$.

$(\Leftarrow)$ Suppose that $(\mathcal{Q},\mathcal{D})^{\circ} = (\mathcal{Q},\mathcal{D})$ to prove $(\mathcal{Q},\mathcal{D})$ is Double I-open. Since $(\mathcal{Q},\mathcal{D})$ is Double I-open set and $(\mathcal{Q},\mathcal{D})^{\circ} = \cup\{(\mathcal{C},\mathcal{G}) \in \Psi; (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D})\} \rightarrow (\mathcal{Q},\mathcal{D})^{\circ}$ is a Double I-open set. Thus $(\mathcal{Q},\mathcal{D})$ is Double I-open iff $(\mathcal{Q},\mathcal{D})^{\circ} = (\mathcal{Q},\mathcal{D})$.

4) $(\tilde{\emptyset},\tilde{\emptyset})^{\circ} = (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow \widetilde{(\emptyset,\emptyset)} \subseteq (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow \widetilde{(\emptyset,\emptyset)} \subseteq (\tilde{\emptyset},\tilde{\emptyset})^{\circ} \subseteq (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow \widetilde{(\emptyset,\emptyset)}^{\circ} = (\tilde{\emptyset},\tilde{\emptyset})$.

$(\tilde{X},\tilde{X})^{\circ} = (\tilde{X},\tilde{X}) \rightarrow (\tilde{X},\tilde{X}) \subseteq (\tilde{X},\tilde{X}) \rightarrow (\tilde{X},\tilde{X}) \subseteq (\tilde{X},\tilde{X})^{\circ} \subseteq (\tilde{X},\tilde{X}) \rightarrow (\tilde{X},\tilde{X})^{\circ} = (\tilde{X},\tilde{X})$.

Since $(\mathcal{Q},\mathcal{D})^{\circ}$ is Double I-open set. Therefore $((\mathcal{Q},\mathcal{D})^{\circ})^{\circ} = (\mathcal{Q},\mathcal{D})^{\circ}$ (from (3) a bevo).

5) Suppose that $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G})$ to prove $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{C},\mathcal{G})^{\circ}$. Let $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{Q},\mathcal{D})$ and $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{C},\mathcal{G})$, but $(\mathcal{C},\mathcal{G})^{\circ}$ is the largest Double I-open set of $(\mathcal{C},\mathcal{G})$ contained in $(\mathcal{C},\mathcal{G})$. Therefore $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{C},\mathcal{G})^{\circ}$. Hence of $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{C},\mathcal{G})^{\circ}$.

6) To prove $(\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ} = ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ}$, we must prove $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ} \subseteq (\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ} \wedge (\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ} \subseteq ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ} \rightarrow (\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D}) \wedge (\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ} \subseteq (\mathcal{Q},\mathcal{D})^{\circ} \wedge ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ} \subseteq (\mathcal{C},\mathcal{G})^{\circ} \rightarrow ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ} \subseteq (\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ} \ldots(1)$

As $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{Q},\mathcal{D})$. $(\mathcal{C},\mathcal{G})^{\circ} \subseteq (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ} \subseteq (\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G})$, since $(\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ}$ is Double I-open sets containing in $(\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G})$ and $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ}$ is the large Double I-open set containing in $(\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}) \rightarrow ((\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ})^{\circ} \subseteq ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ} \rightarrow (\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ} \subseteq ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ}$ ...(2) from (1) and (2), we have $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{\circ} = (\mathcal{Q},\mathcal{D})^{\circ} \cap (\mathcal{C},\mathcal{G})^{\circ}$.

7) As $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}) \wedge (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{Q},\mathcal{D})^{\circ} \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))^{\circ} \wedge (\mathcal{C},\mathcal{G})^{\circ} \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))^{\circ} \rightarrow ((\mathcal{Q},\mathcal{D})^{\circ} \cup (\mathcal{C},\mathcal{G})^{\circ} \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))^{\circ}$.

Remark 4.2 The converse of property (5) is not true i.e., $(\mathcal{Q},\mathcal{D})^{\circ} \subseteq (\mathcal{C},\mathcal{G})^{\circ} \nRightarrow (\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G})$. The following example show that:

Example 4.3 Let $X = \{1,2,3\}$; $\Psi = \{(\tilde{\emptyset},\tilde{\emptyset}), (\tilde{X},\tilde{X}), (\xi_1,\xi_3), (\xi_2,\xi_3), (\xi_3,\xi_3), (\tilde{\emptyset},\xi_3)\}$ where $(\xi_1,\xi_3) = (\langle x,\{1\},\{2,3\}\rangle, \langle x,\{1,2\},\emptyset\rangle)$, $(\xi_2,\xi_3) = (\langle x,\{2\},\{1\}\rangle, \langle x,\{1,2\},\emptyset\rangle)$, $(\xi_3,\xi_3) = (\langle x,\{1,2\},\emptyset\rangle, \langle x,\{1,2\},\emptyset\rangle)$ and $(\tilde{\emptyset},\xi_3) = (\langle x,\emptyset,X\rangle, \langle x,\{1,2\},\emptyset\rangle)$. Let $(\xi_4,\xi_3) = (\langle x,\{1\},\{2\}\rangle, \langle x,\{1,2\},\emptyset\rangle)$, $(\xi_5\xi_3) = (\langle x,\{1\},\{3\}\rangle, \langle x,\{1,2\},\emptyset\rangle)$. Since $(\xi_4,\xi_3)^{\circ} = (\xi_1,\xi_3)$ and $(\xi_5,\xi_3)^{\circ} = (\xi_1,\xi_3)$. Notes that $(\xi_4,\xi_3)^{\circ} \subseteq (\xi_5,\xi_3)^{\circ}$, but $(\xi_4,\xi_3) \not\subset (\xi_5,\xi_3)$.

Remark 4.4 The converse contains of property (7) is not true in general, i.e., $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))^{\circ} \not\subset (\mathcal{Q},\mathcal{D})^{\circ} \cup (\mathcal{C},\mathcal{G})^{\circ}$.

In the previous example: let $(\xi_6,\xi_6) = (\langle x,\{2,3\},\emptyset\rangle, \langle x,\{2.3\},\emptyset\rangle)$, then $(\xi_4,\xi_3) \cup (\xi_6,\xi_6) = (\tilde{X},\tilde{X}) \rightarrow ((\xi_4,\xi_3) \cup (\xi_6,\xi_6))^{\circ} = (\tilde{X},\tilde{X})$. But $(\xi_4,\xi_3)^{\circ} \cup (\xi_6,\xi_6)^{\circ} = (\xi_1,\xi_3)$ and $(\tilde{X},\tilde{X}) \not\subset (\xi_1,\xi_3)$.

Theorem 4.5 Let (X,Ψ) be a DITS and $(\mathcal{Q},\mathcal{D}), (\mathcal{C},\mathcal{G}) \in$ Double I (X). The following characterizations are hold:

1) $(\mathcal{Q},\mathcal{D}) \subseteq \mathrm{cl}\,(\mathcal{Q},\mathcal{D})$.
2) $\mathrm{cl}\,(\mathcal{Q},\mathcal{D})$ is smallest Double I-closed set contains $(\mathcal{Q},\mathcal{D})$.
3) $(\mathcal{Q},\mathcal{D})$ is Double I-closed $\Leftrightarrow \mathrm{cl}\,(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D})$.
4) If $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G})$ then $\mathrm{cl}(\mathcal{Q},\mathcal{D}) \subseteq \mathrm{cl}\,(\mathcal{C},\mathcal{G})$.
5) $\mathrm{cl}\,((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G})) = \mathrm{cl}\,(\mathcal{Q},\mathcal{D}) \cup \mathrm{cl}\,(\mathcal{C},\mathcal{G})$.

P- ISSN 1991-8941 E-ISSN 2706-6703 Journal of University of Anbar for Pure Science (JUAPS) Open Access
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6) cl $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G})) \subseteq$ cl $(\mathcal{Q},\mathcal{D}) \cap$ cl $(\mathcal{C},\mathcal{G})$.

7) cl $(\widetilde{\mathrm{X}}, \widetilde{\mathrm{X}}) = (\widetilde{\mathrm{X}}, \widetilde{\mathrm{X}})$, cl $(\widetilde{\emptyset}, \widetilde{\emptyset}) = (\widetilde{\emptyset}, \widetilde{\emptyset})$ and cl (cl $(\mathcal{Q},\mathcal{D})$)= cl $(\mathcal{Q},\mathcal{D})$.

Proof 1) By definition of Double closure of $(\mathcal{Q},\mathcal{D})$, we have cl$(\mathcal{Q},\mathcal{D})$ is the intersection Double I-closed set containing $(\mathcal{Q},\mathcal{D})$ clearly $(\mathcal{Q},\mathcal{D}) \subseteq$ cl $(\mathcal{Q},\mathcal{D})$.

2) Since the intersection of any number of Double I-closed set is also Double I-closed set, so the Double closure of $(\mathcal{Q},\mathcal{D})$, being the intersection of its Double I-closed contains is a Double I-closed set and containing of $(\mathcal{Q},\mathcal{D})$. Since the intersection of any number of Double I-closed set is always a subset of each of Double I-closed set contain of $(\mathcal{Q},\mathcal{D})$.

3) Suppose that $(\mathcal{Q},\mathcal{D})$ is Double I-closed to prove cl $(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D})$, since $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D}) \rightarrow (\mathcal{Q},\mathcal{D})$ is Double I-closed set containing of $(\mathcal{Q},\mathcal{D})$. But cl $(\mathcal{Q},\mathcal{D})$ is smallest Double I-closed set containing $(\mathcal{Q},\mathcal{D})$ (by (2))→ cl $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D})$ … (1). Also, by property (1) $(\mathcal{Q},\mathcal{D}) \subseteq$ cl $(\mathcal{Q},\mathcal{D})$ ... (2) From (1) and (2), we have cl $(\mathcal{Q},\mathcal{D})= (\mathcal{Q},\mathcal{D})$.

Conversely If $(\mathcal{Q},\mathcal{D})$ = cl $(\mathcal{Q},\mathcal{D})$, since cl $(\mathcal{Q},\mathcal{D})$ Double I-closed set → $(\mathcal{Q},\mathcal{D})$ is also Double I-closed set. Hence the proof.

4) Suppose that $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G})$ and $(\mathcal{C},\mathcal{G}) \subseteq$ cl $(\mathcal{C},\mathcal{G})$ (by (1) $(\mathcal{Q},\mathcal{D}) \subseteq$ cl$(\mathcal{Q},\mathcal{D})$) → $(\mathcal{Q},\mathcal{D}) \subseteq$ cl $(\mathcal{C},\mathcal{G})$ (i.e., cl $(\mathcal{Q},\mathcal{D})$ is Double I-closed set containing $(\mathcal{Q},\mathcal{D})$. But cl $(\mathcal{Q},\mathcal{D})$ is smallest Double I-closed set containing $(\mathcal{Q},\mathcal{D})$. Therefore cl $(\mathcal{Q},\mathcal{D}) \subseteq$ cl $(\mathcal{C},\mathcal{G})$ → if $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G})$, then cl $(\mathcal{Q},\mathcal{D}) \subseteq$ cl $(\mathcal{C},\mathcal{G})$.

5) Let $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G})$ → cl $(\mathcal{Q},\mathcal{D}) \subseteq$ cl $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))$ (by (4)) and $(\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G})$ → cl $(\mathcal{C},\mathcal{G}) \subseteq$ cl

$((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))$ → cl $(\mathcal{Q},\mathcal{D}) \cup$ cl $(\mathcal{C},\mathcal{G}) \subseteq$ cl $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))$ ..(1) We have $(\mathcal{Q},\mathcal{D}) \subseteq$ cl $(\mathcal{Q},\mathcal{D})$ and $(\mathcal{C},\mathcal{G}) \subseteq$ cl $(\mathcal{C},\mathcal{G})$ (by property (1)) → $(\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}) \subseteq$ cl $(\mathcal{Q},\mathcal{D})$ ∪cl $(\mathcal{C},\mathcal{G})$. i.e., cl $(\mathcal{Q},\mathcal{D}) \cup$ cl $(\mathcal{C},\mathcal{G})$ is Double I-closed set containing $(\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G})$, but cl $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))$ is smallest Double I-closed set of $(\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G})$. (by property (2)). Hence cl $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G})) \subseteq$ cl $(\mathcal{Q},\mathcal{D})$ ∪cl $(\mathcal{C},\mathcal{G})$… (2) from (1) and (2), we have cl $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))$= cl $(\mathcal{Q},\mathcal{D}) \cup$ cl $(\mathcal{C},\mathcal{G})$.

6) To show that cl $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G})) \subseteq$ cl $(\mathcal{Q},\mathcal{D}) \cap$ cl $(\mathcal{C},\mathcal{G})$ → $(\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D})$ → cl $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G})) \subseteq$ cl $(\mathcal{Q},\mathcal{D})$ and $(\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{C},\mathcal{G})$ → cl $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G})) \subseteq$ cl $(\mathcal{C},\mathcal{G})$. So cl $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G})) \subseteq$ cl $(\mathcal{Q},\mathcal{D}) \cap$ cl $(\mathcal{C},\mathcal{G})$.

7) Since each of $(\widetilde{\mathrm{X}}, \widetilde{\mathrm{X}})$, $(\widetilde{\emptyset}, \widetilde{\emptyset})$ of cl $(\mathcal{Q},\mathcal{D})$ are Double I-closed set and from (3) property, $(\mathcal{Q},\mathcal{D})$ is Double I-closed set iff cl $(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D})$ → cl $(\widetilde{\mathrm{X}}, \widetilde{\mathrm{X}}) = (\widetilde{\mathrm{X}}, \widetilde{\mathrm{X}})$, cl $(\widetilde{\emptyset}, \widetilde{\emptyset}) = (\widetilde{\emptyset}, \widetilde{\emptyset})$ and cl (cl $(\mathcal{Q},\mathcal{D})$) = cl$(\mathcal{Q},\mathcal{D})$.

Remark 4.6 The converse contains of property (4) is not true in general for example:

Example 4.7 Let $\mathrm{X} = \{g, h, j\}$; $\Psi = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{\mathrm{X}},\widetilde{\mathrm{X}}), (\lambda_1,\lambda_2), (\lambda_1,\lambda_3), (\lambda_1,\lambda_4), (\lambda_1,\lambda_1)\}$ where $(\lambda_1,\lambda_2) = (\langle x,\{g\},\{h\}\rangle, \langle x,\{g,j\},\{h\}\rangle)$, $(\lambda_1,\lambda_3) = (\langle x,\{g\},\{h\}\rangle, \langle x,\{g\},\emptyset\rangle)$, $(\lambda_1,\lambda_4) = (\langle x,\{g\},\{h\}\rangle, \langle x,\{g,j\},\emptyset\rangle)$ and $(\lambda_1,\lambda_1) = (\langle x,\{g\},\{h\}\rangle, \langle x,\{g\},\{h\}\rangle)$. $\Psi^c = \{(\widetilde{\emptyset}, \widetilde{\emptyset}), (\widetilde{\mathrm{X}}, \widetilde{\mathrm{X}}), (\lambda_2^c,\lambda_1^c), (\lambda_3^c,\lambda_1^c), (\lambda_4^c,\lambda_1^c), (\lambda_1^c,\lambda_1^c)\}$ where $(\lambda_2^c,\lambda_1^c) = (\langle x,\{h\},\{g,j\}\rangle, \langle x,\{h\},\{g\}\rangle)$, $(\lambda_3^c,\lambda_1^c) = \langle x,\emptyset,\{g\}\rangle, \langle x,\{h\},\{g\}\rangle)$, $(\lambda_4^c,\lambda_1^c) = (\langle x,\emptyset,\{g,j\}\rangle, \langle x,\{h\},\{g\}\rangle)$ *and* $(\lambda_1^c,\lambda_1^c) = (\langle x,\{h\},\{g\}\rangle, \langle x,\{h\},\{g\}\rangle)$. Let (λ_1,λ_1) and $(\lambda_5,\lambda_6) = (\langle x,\{j\},\{g,h\}\rangle, \langle x,\{j\},\{h\}\rangle)$. Notes that cl$(\lambda_1,\lambda_1) \subseteq$ cl (λ_5,λ_6) but $(\lambda_1,\lambda_1) \not\subset (\lambda_5,\lambda_6)$.

Remark 4.8 The converse of property (6) is not true: i.e., cl $(\mathcal{Q},\mathcal{D}) \cap$ cl $\mathcal{C},\mathcal{G}) \not\subset$ cl $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))$.

Example 4.9 Let $\mathrm{X} = \{a, b, c\}$; $\Psi = \{(\widetilde{\emptyset}, \widetilde{\emptyset}), (\widetilde{\mathrm{X}}, \widetilde{\mathrm{X}}), (s_2,s_1), (\widetilde{\emptyset},s_1), (\widetilde{\emptyset},s_3)\}$ where $(\widetilde{\emptyset},s_1) = (\langle x,\emptyset,\mathrm{X}\rangle, \langle x,\{a,b\},\emptyset\rangle)$, $(s_2,s_1) = (\langle x,\{a\},\{b\}\rangle, \langle x,\{a,b\},\emptyset\rangle)$, $(\widetilde{\emptyset},s_3) = (\langle x,\emptyset,\mathrm{X}\rangle, \langle x,\{a,b\},\{c\}\rangle)$. $\Psi^c = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{\mathrm{X}},\widetilde{\mathrm{X}}), (s_1^c,s_2^c), (s_1^c,\widetilde{\mathrm{X}}), (s_3^c,\widetilde{\mathrm{X}})\}$ where $(s_1^c,s_2^c)= \langle x,\emptyset,\{a,b\}\rangle, \langle x,\{b\},\{a\}\rangle)$, $(s_1^c,\widetilde{\mathrm{X}}) = (\langle x,\emptyset,\{a,b\}\rangle, \langle x,\mathrm{X},\emptyset\rangle)$, $(s_3^c,\widetilde{\mathrm{X}}) = \langle x,\{c\},\{a,b\}\rangle, \langle x,\mathrm{X},\emptyset\rangle)$. Let $(s_4,s_5) = (\langle x,\{c\},\{b\}\rangle, \langle x,\{a,c\},\{b\}\rangle)$ and let $(s_2^c,s_1) = (\langle x,\{b\},\{a\}\rangle, \langle x,\{a,b\},\emptyset\rangle)$. Clear $(s_4,s_5) \cap (s_2^c,s_1)= (\langle x,\emptyset,\{a,b\}\rangle, \langle x,\{a\},\{b\}\rangle)$ → cl $((s_4,s_5) \cap (s_2^c,s_1)) = (s_1^c,\widetilde{\mathrm{X}})$. But cl $(s_4,s_5) \cap cl\ (s_2^c,s_1) = (\widetilde{\mathrm{X}},\widetilde{\mathrm{X}})$.

Hence cl $(s_4,s_5) \cap cl\ (s_2^c,s_1) \not\subset$ cl $((s_4,s_5)$

$\cap (s_2^c, s_1))$.

Theorem 4.10 Let $(\text{Ҳ}, \Psi)$ be a DITS and $(\mathcal{Q},\mathcal{D})$, $(\mathcal{C},\mathcal{G}) \in$ Double I (Ҳ). The following characterizations are hold:

1) $(\mathcal{Q},\mathcal{D})$ is Double I-closed $\Leftrightarrow (\mathcal{Q},\mathcal{D})' \subseteq (\mathcal{Q},\mathcal{D})$.
2) $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{Q},\mathcal{D})' \subseteq (\mathcal{C},\mathcal{G})'$.
3) $(\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})'$ is Double I-closed.
4) $((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))' \subseteq (\mathcal{Q},\mathcal{D})' \cap (\mathcal{C},\mathcal{G})'$.
5) $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))' = (\mathcal{Q},\mathcal{D})' \cup (\mathcal{C},\mathcal{G})'$.

Proof 1) Suppose that $(\mathcal{Q},\mathcal{D})$ is Double I-closed to prove that $(\mathcal{Q},\mathcal{D})' \subseteq (\mathcal{Q},\mathcal{D})$, so $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \notin (\mathcal{Q},\mathcal{D}) \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})^c$ and $(\mathcal{Q},\mathcal{D})^c$ Double I-open, $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})^c$, $(\mathcal{Q},\mathcal{D})^c \cap (\mathcal{Q},\mathcal{D}) = (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})' \rightarrow (\mathcal{Q},\mathcal{D})' \subseteq (\mathcal{Q},\mathcal{D})$.

Conversely Suppose that $(\mathcal{Q},\mathcal{D})' \subseteq (\mathcal{Q},\mathcal{D})$ to prove $(\mathcal{Q},\mathcal{D})$ is Double I-closed, since $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})^c \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \notin (\mathcal{Q},\mathcal{D}) \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \notin (\mathcal{Q},\mathcal{D})'$ there exist Double I-open set (υ,ν) such that $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\upsilon,\nu)$ and $(\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap (\mathcal{Q},\mathcal{D}) = (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow (\upsilon,\nu) \cap (\mathcal{Q},\mathcal{D}) = (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow (\upsilon,\nu) \subseteq (\mathcal{Q},\mathcal{D})^c \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\upsilon,\nu) \subseteq (\mathcal{Q},\mathcal{D})^c \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in \text{int}\,(\mathcal{Q},\mathcal{D})^c$. So $(\mathcal{Q},\mathcal{D})^c \subseteq \text{int}\,(\mathcal{Q},\mathcal{D})^c$, we know that $\text{int}\,(\mathcal{Q},\mathcal{D})^c \subseteq (\mathcal{Q},\mathcal{D})^c \rightarrow \text{int}\,(\mathcal{Q},\mathcal{D})^c = (\mathcal{Q},\mathcal{D})^c$ is Double I-open ((by theorem 4.1(3)), $(\mathcal{Q},\mathcal{D})$ is Double I-open $\Leftrightarrow (\mathcal{Q},\mathcal{D})^o = (\mathcal{Q},\mathcal{D})$. Therefore $(\mathcal{Q},\mathcal{D})$) is Double I-closed.

2) Since $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})^c \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})$ is Double limit point $\rightarrow \exists$ Double I-open containing $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})$ and contains at least one Double I-point set of $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{C},\mathcal{G})' \rightarrow (\mathcal{Q},\mathcal{D})' \subseteq (\mathcal{C},\mathcal{G})'$.

3) i.e., $(\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})'$ is Double I-closed $\rightarrow ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D}))^c$ is Double I-open to prove that $\text{int}\,((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c = ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c$ so $\text{int}\,((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c$…(1)

To prove that $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c \subseteq \text{int}\,((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c$… (2). Let $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \notin (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})' \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \notin (\mathcal{Q},\mathcal{D})$ and $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})'$ $\exists$ Double I-open set (υ,ν) such that $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\upsilon,\nu)$, $(\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap (\mathcal{Q},\mathcal{D}) = (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow (\upsilon,\nu) \cap (\mathcal{Q},\mathcal{D}) = (\tilde{\emptyset},\tilde{\emptyset})$. $(\upsilon,\nu) \cap (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})' = (\upsilon,\nu) \cap (\mathcal{Q},\mathcal{D}) \cup (\upsilon,\nu) \cap (\mathcal{Q},\mathcal{D})' = (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow (\upsilon,\nu) \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c$ and $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in \text{int}\,((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c$.
$((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c \subseteq \text{int}((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c \rightarrow ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c = \text{int}\,((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c \rightarrow ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})')^c$ is Double I-open set $\rightarrow (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})'$ is Double I-closed set .

4)Since $(\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D}) \rightarrow ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))' \subseteq (\mathcal{Q},\mathcal{D})'$, $(\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}) \subseteq (\mathcal{C},\mathcal{G}) \rightarrow ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))' \subseteq (\mathcal{C},\mathcal{G})' \rightarrow ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))' \subseteq (\mathcal{Q},\mathcal{D})' \cap (\mathcal{C},\mathcal{G})'$.

5) $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{Q},\mathcal{D})' \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))'$, $(\mathcal{C},\mathcal{G}) \subseteq (\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}) \rightarrow (\mathcal{C},\mathcal{G})' \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))' \rightarrow (\mathcal{Q},\mathcal{D})' \cup (\mathcal{C},\mathcal{G})' \subseteq ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))'$ … (1)

to prove that $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))' \subseteq (\mathcal{Q},\mathcal{D})' \cup (\mathcal{C},\mathcal{G})'$. Let $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))' \rightarrow \exists$ Double I-open set (υ,ν) such that $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\upsilon,\nu) \wedge (\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap ((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G})) \neq (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow (\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap (\mathcal{Q},\mathcal{D}) \cup (\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap (\mathcal{C},\mathcal{G})) \neq (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow (\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap (\mathcal{Q},\mathcal{D}) \neq (\tilde{\emptyset},\tilde{\emptyset})$ or $(\upsilon,\nu) \setminus \{(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}})\} \cap$

$(\mathcal{Q},\mathcal{D}) \neq (\tilde{\emptyset},\tilde{\emptyset})$, so $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})'$ or $(\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{C},\mathcal{G})' \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})' \cup (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{C},\mathcal{G})' \rightarrow (\tilde{\mathfrak{p}},\tilde{\mathfrak{p}}) \in (\mathcal{Q},\mathcal{D})' \cup$

$(\mathcal{C},\mathcal{G})'$, hence $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))' \subseteq (\mathcal{Q},\mathcal{D})' \cup (\mathcal{C},\mathcal{G})'$…(2) from (1) and (2), we have $((\mathcal{Q},\mathcal{D}) \cup (\mathcal{C},\mathcal{G}))' = (\mathcal{Q},\mathcal{D})' \cup (\mathcal{C},\mathcal{G})'$.

Remark 4.11 The converse of property (2) is not true: i.e., $(\mathcal{Q},\mathcal{D})' \subseteq (\mathcal{C},\mathcal{G})' \nRightarrow (\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{C},\mathcal{G})$. For example:

Example 4.12 Let $\text{Ҳ} = \{v, k, o\}$; $\Psi = \{(\tilde{\emptyset},\tilde{\emptyset}), (\tilde{\text{Ҳ}},\tilde{\text{Ҳ}}), (L_1, L_2), (L_3, L_4), (L_4, \tilde{\text{Ҳ}}), (L_5, L_1)\}$ where $(L_1, L_2) = (\langle x, \{o\}, \{v,k\}\rangle, \langle x, \{v,o\}, \{k\}\rangle)$, $(L_3, L_4) = (\langle x, \{k\}, \{v\}\rangle, \langle x, \{k,o\}, \{v\}\rangle)$, $(L_4, \tilde{\text{Ҳ}}) = (\langle x, \{k,o\}, \{v\}\rangle, \langle x, \text{Ҳ}, \emptyset\rangle)$ and $(L_5, L_1) = (\langle x, \emptyset, \{v,k\}\rangle, \langle x, \{o\}, \{v,k\}\rangle)$. Let $(L_6, L_4) = (\langle x, \{o\}, \{v\}\rangle, \langle x, \{k,o\}, \{v\}\rangle)$ and $(L_2, L_2) = (\langle x, \{v,o\}, \{k\}\rangle, \langle x, \{v,o\}, \{k\}\rangle)$ to find the Double limit point of any Double I-set must choose every Double I-open sets for every Double I-point in Ҳ and notes satisfy the definition or not.

$(\tilde{v},\tilde{v}) = (\langle x,\{v\},\{k,o\}\rangle,\langle x,\{v\},\{k,o\}\rangle) \in \mathbb{X}$ and the Double I-set containing $(\tilde{v},\tilde{v})$ is $(\tilde{\mathbb{X}},\tilde{\mathbb{X}})$ only.

$(\tilde{k},\tilde{k}) = (\langle x,\{k\},\{v,o\}\rangle,\langle x,\{k\},\{v,o\}\rangle) \in \mathbb{X}$ and the Double I-set containing $(\tilde{k},\tilde{k})$ are $(\tilde{\mathbb{X}},\tilde{\mathbb{X}})$, (L_3,L_4) and $(L_4,\tilde{\mathbb{X}})$.

$(\tilde{o},\tilde{o}) = (\langle x,\{o\},\{v,k\}\rangle,\langle x,\{o\},\{v,k\}\rangle) \in \mathbb{X}$ and the Double I-set containing $(\tilde{o},\tilde{o})$ are $(\tilde{\mathbb{X}},\tilde{\mathbb{X}})$, (L_1,L_2) and $(L_4,\tilde{\mathbb{X}})$. Notes that $(\tilde{\mathbb{X}},\tilde{\mathbb{X}}) \setminus \{((\tilde{v},\tilde{v})\} \cap (L_6,L_4) = (L_6,L_4) \to (\tilde{v},\tilde{v}) \in (L_6,L_4)^{/}$.

Notes that $(L_3,L_4) \setminus \{((\tilde{k},\tilde{k})\} \cap (L_6,L_4) = (\langle x,\emptyset,\{v,k\}\rangle,\langle x,\{o\},\{v,k\}\rangle) \cap (L_6,L_4) = (L_5,L_1)$.

Notes that $(L_4,\tilde{\tilde{v}})\setminus \{((\tilde{k},\tilde{k})\} \cap (L_6,L_4) = (\langle x,\{o\},\{v,k\}\rangle,\langle x,\{o\},\{v,k\}\rangle) \cap (L_6,L_4) = (L_1,L_1) \to (\tilde{k},\tilde{k}) \in (L_6,L_4)^{/}$. Notes that $(\tilde{\mathbb{X}},\tilde{\mathbb{X}}) \setminus \{(\tilde{o},\tilde{o})\} \cap (L_6,L_4) = (L_8,L_9)$. Notes that $(L_1,L_2) \setminus \{(\tilde{o},\tilde{o})\} \cap (L_6,L_4) = (\tilde{\emptyset},\tilde{\emptyset}) \to (\tilde{o},\tilde{o}) \notin (L_6,L_4)^{/}$. Therefore $(L_6,L_4)^{/} = \{(\tilde{v},\tilde{v}),(\tilde{k},\tilde{k})\}$.

To find $(L_2,L_2)^{/}$. Notes that $(\tilde{\mathbb{X}},\tilde{\mathbb{X}}) \setminus \{((\tilde{v},\tilde{v})\} \cap (L_2,L_2) = (L_1,L_1) \to (\tilde{v},\tilde{v}) \in (L_2,L_2)^{/}$.

Notes that $(\tilde{\mathbb{X}},\tilde{\mathbb{X}}) \setminus \{((\tilde{k},\tilde{k})\} \cap (L_2,L_2) = (L_2,L_2)$ .Notes that $(L_3,L_4) \setminus \{((\tilde{k},\tilde{k})\} \cap (L_2,L_2) = (L_5,L_1)$.

Notes that $(L_4,\tilde{\mathbb{X}}) \setminus \{((\tilde{k},\tilde{k})\} \cap (L_2,L_2) = (L_1,L_2) \cap (L_2,L_2) = (L_1,L_2) \to (\tilde{k},\tilde{k}) \in (L_2,L_2)^{/}$.

Notes that $(\tilde{\mathbb{X}},\tilde{\mathbb{X}}) \setminus \{(\tilde{o},\tilde{o})\} \cap (L_2,L_2) = (L_4^c,L_4^c)$. Notes that $(L_1,L_2) \setminus \{(\tilde{o},\tilde{o})\} \cap (L_2,L_2) = (\tilde{\emptyset},L_4^c)$.

Notes that $(L_4,\tilde{\mathbb{X}}) \setminus \{((\tilde{o},\tilde{o})\} \cap (L_2,L_2) = (\tilde{\emptyset},L_4^c) \to (\tilde{o},\tilde{o}) \in (L_2,L_2)^{/}$. Therefore $(L_2,L_2)^{/} = \{(\tilde{\mathbb{X}},\tilde{\mathbb{X}})\}$. Hence $(L_6,L_4)^{/} \subseteq (L_2,L_2)^{/}$ but $(L_6,L_4) \not\subset (L_2,L_2)$.

Remark 4.13 The converse of property (4) is not true in general i.e., $(\mathcal{Q},\mathcal{D})^{/} \cap (\mathcal{C},\mathcal{G})^{/} \not\subset ((\mathcal{Q},\mathcal{D}) \cap (\mathcal{C},\mathcal{G}))^{/}$:

Example 4.14 Recall Example 4.12, we see that $(L_2,L_2)^{/} \cap (L_2^c,L_2^c)^{/} \not\subset ((L_2,L_2) \cap (L_2^c,L_2^c))^{/}$.

Let $(L_2^c,L_2^c) = (\langle x,\{k\},\{v,o\}\rangle,\langle x,\{k\},\{v,o\}\rangle)$ to find the Double limit point of any Double I-set. Notes that $(\tilde{\mathbb{X}},\tilde{\mathbb{X}}) \setminus \{((v,\tilde{v})\} \cap (L_2^c,L_2^c) = (L_2^c,L_2^c) \to (\tilde{v},\tilde{v}) \in (L_2^c,L_2^c)^{/}$.

Notes that $(\tilde{\mathbb{X}},\tilde{\mathbb{X}}) \setminus \{((\tilde{k},\tilde{k})\} \cap (L_2^c,L_2^c) = (\tilde{\emptyset},\tilde{\emptyset}) \to (\tilde{k},\tilde{k}) \notin (L_2^c,L_2^c)^{/}$. Notes that $(\tilde{\mathbb{X}},\tilde{\mathbb{X}}) \setminus \{(\tilde{o},\tilde{o})\} \cap (L_2^c,L_2^c) = (L_2^c,L_2^c)$.

Notes that $(L_1,L_2) \setminus \{(\tilde{o},\tilde{o})\} \cap (L_2^c,L_2^c) = (\tilde{\emptyset},\tilde{\emptyset}) \to (\tilde{o},\tilde{o}) \notin (L_2^c,L_2^c)^{/}$. Therefore $(L_2^c,L_2^c)^{/} = \{(\tilde{v},\tilde{v})\}$ and $(L_2,L_2)^{/} = \{(\tilde{\mathbb{X}},\tilde{\mathbb{X}})\}$, then $(L_2,L_2)^{/} \cap (L_2^c,L_2^c)^{/} = \{(\tilde{v},\tilde{v})\}$ to find $((L_2,L_2) \cap (L_2^c,L_2^c))^{/}$, since $((L_2,L_2) \cap (L_2^c,L_2^c)) = (\tilde{\emptyset},\tilde{\emptyset})$. By remark 3.7 (1) if $(\mathcal{Q},\mathcal{D})) = (\tilde{\emptyset},\tilde{\emptyset}) \subseteq \mathbb{X} \to (\tilde{\emptyset},\tilde{\emptyset})^{/} = (\tilde{\emptyset},\tilde{\emptyset})$. Hence $((L_2,L_2) \cap (L_2^c,L_2^c))^{/} = (\tilde{\emptyset},\tilde{\emptyset})$.

Theorem 4.15 Let $(\mathbb{X},\Psi)$ be a DITS and $(\mathcal{Q},\mathcal{D}) \in$ Double I $(\mathbb{X})$. Then cl $(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/}$.

Proof To prove that cl $(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/}$, we must prove cl $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/} \wedge (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/} \subseteq$ cl $(\mathcal{Q},\mathcal{D})$. Since $(\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/}$ is Double I-closed ((by theorem 4.10 (3)))

And containing $(\mathcal{Q},\mathcal{D})$, cl $(\mathcal{Q},\mathcal{D}) \subseteq (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/}$…. (1). To prove $(\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/} \subseteq$ cl $(\mathcal{Q},\mathcal{D})$. Since $(\mathcal{Q},\mathcal{D}) \subseteq$ cl $(\mathcal{Q},\mathcal{D}) \to (\mathcal{Q},\mathcal{D})^{/} \subseteq$ (cl $(\mathcal{Q},\mathcal{D}))^{/} \subseteq$ cl $(\mathcal{Q},\mathcal{D})$ (by theorem 4.10 (1)) $\to (\mathcal{Q},\mathcal{D})^{/} \subseteq$ cl $(\mathcal{Q},\mathcal{D}) \to (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/} \subseteq$ cl$(\mathcal{Q},\mathcal{D})$… (2) from (1) and (2), we have cl $(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D}) \cup (\mathcal{Q},\mathcal{D})^{/}$.

5 - CONCLUSIONS:

In this paper, we got the following results:

1) We have introduced a new set of the following concepts: Double intuitionistic set (Double IS) (resp., Double intuitionistic topological spaces (DITS), Double I-point, Double I-interior set, Double I closure set and Double I limit point) in DITS .
2) Study the basic characteristics and qualities related to these types and the relationships between them and giving examples is incorrect.

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المجموعات المفتوحة الحدسية المزدوجة مع بعض التطبيقات

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الخلاصة:

الغرض من هذا العمل هو تقديم فئة جديدة من المجموعات المفتوحة وهي المجموعات المفتوحة الحدسية المزدوجة. تمت دراسة العلاقات بين المجموعات المفتوحة الحدسي المزدوج والمجموعات الحدسية المزدوجة بما في ذلك المجموعة الداخلية الحدسية المزدوجة ومجموعة الإغلاق الحدسي المزدوج ونقطة الغاية الحدسية المزدوجة في المساحات التبولوجية الحدسية المزدوجة وتم تقديم أمثلة مختلفة والعديد من الملاحظات لكل مفهوم ، أيضًا تم تقديم تعريفات المجموعة المزدوجة في فضاءات تبولوجية عامة لذلك ، قمنا بتعميمها على المساحات التبولوجية الحدسية المزدوجة مع تقديم النظرية الأساسية لهذا الفضاء الجديد ، والعديد من الصفات والخصائص الأساسية المتعلقة بهذه المفاهيم التي يتم تقديمها في القسم الثالث كدليل مع العديد من الأمثلة مع مراعاة أن العكس ليس صحيحًا لكل دليل على هذه الخصائص التي تم عرضها وما نلاحظه في القسم الرابع .